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Miniature geometries

BURTON W. JONES, *University of Colorado, Boulder, Colorado.*

Recent interest in emphasizing "patterns" in high school mathematics may lead some teachers to consider miniature geometries as suitable for a unit for gifted students.

IN A PREVIOUS ARTICLE[1] miniature number systems were developed—that is, number systems which have most of the usual properties of numbers with which we are familiar but which contain only a finite number of numbers. Similarly, there are geometries that are "miniature" in the sense that they contain only a finite number of points. Here, however, we are in the beginning forced to make a number of choices: Is it to be plane geometry, solid geometry, or a geometry of many dimensions? Is it to be Euclidean, projective, or any one of the other geometries we know about?

Our first arbitrary choice is to confine ourselves here to plane geometry. That being the case, the following requirements are quite natural:

1. The geometry consists of a set of undefined elements called *points*.
2. It contains certain subsets (smaller sets) of points called *lines*.
3. If L is a line and p is a point in the set of points which comprise L, we call p "a point of L" or write "p lies on L." The same idea is expressed by writing that "L contains p" or "L passes through p."
4. Any two points "determine a line"; that is, given any two points p and q, there is exactly one line L passing through these two points.

Notice that in the first three statements we set up a terminology. This is especially true of the third statement. We could merely write "L contains p" or "p is contained in L," but if we are to get help from geometrical visualization (and this is our chief guide in this development) it is useful to preserve as much of the geometrical terminology as possible. Property 4, on the other hand, is the first really restrictive requirement.

It is natural to impose two other requirements, the first to make it a finite geometry and the second to keep it from being too trivial:

5. The geometry contains only a finite number of points.
6. The geometry contains four points, no three of which lie on a line.

Now we are faced with our second choice. If the geometry were to be Euclidean[2] we would specify:

E7. If L is a line and p a point not on L, there is exactly one line L_p through p which has no points in common with L. L_p is called "parallel to L."

On the other hand, if the geometry were to be projective we would specify:

P7. Every pair of lines has a common point.

[1] Burton W. Jones, "Miniature Number Systems," THE MATHEMATICS TEACHER, LI (April 1958), 226–231. This will hereafter be referred to as "Article I," and its contents will be presupposed in much of this article.

[2] R. H. Bruck, "Recent Advances in the Foundations of Euclidean Plane Geometry," *The American Mathematical Monthly*, 62 (August–September, II, 1955), 2–17.

In this article we make the second choice since it is simpler in a number of respects; if we are to manufacture a geometry we might as well make it as simple as possible. Notice that Property 4 with Property P7 affirms that *any pair of distinct lines has exactly one common point.*

A FINITE GEOMETRY WITH SEVEN POINTS

Before considering the general theory, let us look at one particular finite geometry: one with seven points and seven lines. (This is actually the "smallest" geometry.) Number the points 0, 1, 2, 3, 4, 5, 6 and the seven lines may be taken as the following seven sets of three points each: L_0: 0, 1, 3; L_1: 1, 2, 4; L_2: 2, 3, 5; L_3: 3, 4, 6; L_4: 4, 5, 0; L_5: 5, 6, 1; L_6: 6, 0, 2. Inspection shows that each pair of lines has exactly one common point and that each pair of points determines a line. Also each point has three lines through it, and each line contains three points. This can be seen from the sets of points above or from the diagram.

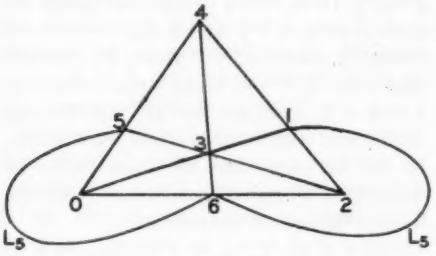

Figure 1

Notice that one of the lines, L_5, is not straight, but recall that the lines are sets of points and the only purpose of drawing the lines is to indicate the sets of points which comprise them.

COUNTING LINES AND POINTS

More generally, suppose one line contains $m+1$ points. First we shall show that every line contains exactly $m+1$ points. Let L_0 be the given line and L any other line. From Property 6 there will be some point p not on either line. Let p_0 be the point common to L_0 and L and p_1, p_2, $\cdots$, p_m the other points on L_0. Then the m lines determined by p and the points on L_0, except p_0, will intersect L in m distinct points. Thus L has at least as many points as L_0. By reversing the argument we can show that L_0 has at least as many points as L and hence both lines have the same number of points.

Next we can show that $m+1$ is the number of lines through a point. Given a point p and some line L not through p, it follows that L has $m+1$ points and that each of these with p determines a line through p. But any line through p intersects L in one of these points. Hence there are just as many lines through p as there are points on L, that is, there are $m+1$ lines through p.

Third, we can count the number of points in the geometry. Given a point p, every point of the geometry will lie on a line through p. Each such line contains m points besides p and there are $m+1$ such lines. Hence the total number of points is $m(m+1)+1=m^2+m+1$.

In a similar fashion it can be shown that there are m^2+m+1 lines in the geometry. Notice that 7 is the least number of lines in a finite geometry satisfying the properties listed above since that is the value of m^2+m+1 when $m=2$ and if $m=1$ there would be only three points in the geometry denying Property 6.

DUALITY

Perhaps the most important consequence of our choice in Property P7 is the *principle of duality:* any true statement about lines and points is also true when "line" and "point" are interchanged and the corresponding change in connective used. For instance, "any line has the same number of points as any other line" implies, by duality, "any point lies on the same number of lines as any other point." This principle seems like something too general to establish until we realize that by virtue of the fact that we assume only Properties

1 to 6 and P7, we need merely verify duality for these seven properties. This can easily be done.

Suppose we consider "another" geometry defined by interchanging "point" and "line" in the Properties 1 through 6 and P7, that is, a dual geometry. We now show that this will have all the properties of the given geometry. In this dual geometry, a point would be identified with the set of lines through it. Property 3 is its own dual, and Property P7 is the dual of Property 4 if included in P7 is the italicized statement at the end of the first section. If the geometry contains only a finite number of lines, it must contain only a finite number of pairs of lines, and hence a finite number of points.

It remains to consider the dual of Property 6, namely 6d: the geometry contains four lines, no three of which have a common point. To show that this implies Property 6, let P_{ab} be the point determined by lines a and b, P_{bc} the point determined by lines b and c, and similarly define P_{cd} and P_{da}. Suppose P_{ab}, P_{bc} and P_{cd} were collinear. The first two of these lie on the line b and the second two on the line c; moreover they *determine* the lines b and c, respectively. But b and c are distinct lines and hence P_{ab}, P_{bc} and P_{cd} are not collinear. In similar fashion any three of the four designated points can be shown to be noncollinear. Thus we have shown that a geometry which satisfies the dual properties, satisfies the given ones. In similar fashion one may show that a geometry which satisfies the given properties, satisfies the dual ones. Hence the two geometries are the same, and the principle of duality is established.

Construction of finite geometries for $m=p^n$

First, let us illustrate the construction for $p=3$, $n=1$, $m=3$. Associate with each point an ordered triple (a, b, c) where a, b, c are numbers of GF(3) (see Article I), not all are zero, and the triples (a, b, c) and (ka, kb, kc) are associated with the same point. For example, (1, 1, 0) and (−1, −1, 0) represent the same point. The number of triples excluding (0, 0, 0) is $3^3-1=26$ and the number of proportionality factors different from zero is $3-1=2$. Hence the number of points in this geometry is $26/2=13$. Notice that $13=m^2+m+1$ for $m=3$. Then the following triples will represent the points of this geometry:

P_0: (0, 0, 1); P_1: (0, 1, 0); P_2: (1, 0, 0); P_3: (0, 1, −1); P_4: (1, −1, 0); P_5: (−1, 1, −1); P_6: (1, 1, 1); P_7: (1, −1, −1); P_8: (−1, 0, −1); P_9: (0, 1, 1); P_{10}: (1, 1, 0); P_{11}: (1, 1, −1); P_{12}: (1, 0, −1).

Now we must determine the lines. Let the point P be associated with the ordered triple $t=(a, b, c)$ and the point P' with $t'=(a', b', c')$. Then the line determined by P and P' will be defined to be all those points associated with the triples $rt+r't'$, that is

$$(ra + r'a', \quad rb + r'b', \quad rc + r'c')$$

where r and r' range over the numbers of GF(3) excluding $r=0=r'$. Thus, if $P=P_0$, $P'=P_1$, $r=0$, $r'=1$ yields the point P_0 above; $r=1$, $r'=0$ yields P_1; $r'=1=-r$ yields P_3, and $r=1=r'$ yields P_9. (Notice that $r=-1$, $r'=1$, being proportional to 1 and −1, yield no new point.) We call L_0 the line composed of these four points. It can be seen that the quadruples of subscripts of points associated with the lines of this geometry will be:

L_0: (0, 1, 3, 9); L_1: (1, 2, 4, 10); L_2: (2, 3, 5, 11); L_3: (3, 4, 6, 12); L_4: (4, 5, 7, 0); L_5: (5, 6, 8, 1); L_6: (6, 7, 9, 2); L_7: (7, 8, 10, 3); L_8: (8, 9, 11, 4); L_9: (9, 10, 12, 5); L_{10}: (10, 11, 0, 6); L_{11}: (11, 12, 1, 7); L_{12}: (12, 0, 2, 8).

It can be shown without much trouble that this geometry satisfies the required properties of a finite geometry with $m=3$.

Similarly if $m=2$, there will be seven points determined by the triples:

P_0: (0, 0, 1); P_1: (0, 1, 0); P_2: (1, 0, 0); P_3: (0, 1, 1); P_4: (1, 1, 0); P_5: (1, 1, 1); P_6: (1, 0, 1).

The lines will be those given above, and the numbers of the triples will be in GF(2).

In general, then for $m=p^n$ we consider the triples (a, b, c) where a, b, c range over the numbers in GF(m), not all are zero, and two triples in which a, b, c are proportional are associated with the same point. Then there will be m^3-1 triples not all zero and $m-1$ proportionality factors different from zero; hence $(m^3-1)/(m-1)=m^2+m+1$ points. The lines are defined as sets of points, as in the above example. This system can be shown to have all the required properties of a finite geometry.

Determination of lines by means of Galois Fields

The reader may have noticed a curious pattern in the two examples above for $m=2$ and $m=3$ in the formation of the successive lines. For instance, for $m=2$, the line L_0 is the set of points P_0, P_1, P_3 and the line L_1 is P_1, P_2, P_4. The subscripts of the points of the latter are those of the former increased by 1. The subscripts of the points in the line L_2 are 2, 3, 5, those for L_3 are 3, 4, 6 and for L_4 are 4, 5, 0 where $6+1$ is replaced by 0, addition being (mod 7). The same pattern holds in the geometry with $m=3$.

Why does this pattern hold? To answer this for $m=2$, observe in a table given at the end of Article I, that x is a generator of GF(8), that is, each power of x will be of the form ax^2+bx+c (mod 2, x^3+x+1). Furthermore, every number of GF (8) will be a power of x. Thus x^i will be expressible in the form $a_ix^2+b_ix+c_i$ where a_i, b_i, c_i are in GF(2). Hence we can associate x^i with the point whose triple is (a_i, b_i, c_i). Thus $x^0=0\cdot x^2+1$ is associated with P_0: (0, 0, 1), $x=0\cdot x^2+1\cdot x+0$ is associated with P_1: (0, 1, 0), $\cdots$, $x^4=x^2+x+0$ with P_4: (1, 1, 0), etc. Since x is also a generator of GF(27), a similar development holds for $m=3$.

Now suppose that x is a generator of GF(m^3) and consider the following three triples:

$t_i=(a_i, b_i, c_i)$; $t_j=(a_j, b_j, c_j)$; $t_k=(a_k, b_k, c_k)$,

where the a's, b's and c's are numbers in GF(m). Suppose the points represented by these triples lie on a line, that is:

$$a_k=ra_i+sa_j,\ b_k=rb_i+sb_j,\ c_k=rc_i+sc_j,$$

where r and s are numbers of GF(m) and not both are zero. Then $x^i=a_ix^2+b_ix+c_i$ with similar expressions for k and j imply $x^k=rx^i+sx^j$ and thus, for every integer u, $x^{k+u}=rx^{i+u}+sx^{j+u}$. Hence $a_{k+u}=ra_{i+u}+sa_{j+u}$ and similarly for b and c. Thus $t_{k+u}=rt_{i+u}+st_{j+u}$ and the points P_{k+u}, P_{i+u}, and P_{j+u} are collinear.

Thus we have shown why the lines are formed from the initial one by the process described. There is one further question which should be answered: Since GF(m^3) contains m^3 numbers, there will be m^3 different powers of x but only m^2+m+1 different points; what is the explanation of this apparent discrepancy? The answer to this question is left to the interested reader.

In general, then, to form a co-ordinate system with co-ordinates in GF(m), we may find a generator of GF(m^3) (not necessarily x as above) whose powers are all quadratic polynomials with coefficients in GF(m). We use the coefficients of the polynomials corresponding to successive powers of the generator, up to the power m^2+m+1, as the co-ordinates of the successive points of our geometry. This elegant device is due to James Singer.[3]

Perfect difference sets

Another approach to the problem of designating the points constituting the various lines of a finite geometry is by way of arithmetical properties of the set of subscripts of the points of line L_0. Consider the numbers 0, 1, 3 in the number system (mod 7). The six differences are: $1-0=1$, $0-1=6$, $3-1=2$, $1-3=5$, $3-0=3$, $0-3=4$ (mod 7). These are just the

[3] James Singer, "A Theorem in Finite Projective Geometry and Some Applications to Number Theory," *Transactions American Mathematical Society*, 43 (1938), 377–385.

numbers 1, 2, 3, 4, 5, 6. In general, a set of numbers 0, 1, 2, $\cdots$, m whose differences (mod m^2+m+1) are 1, 2, 3, $\cdots$, m^2+m in some order is called a *perfect difference set*. Whenever such a set can be found, it can be used as the subscripts for a set of points on a line which may be called L_0. (For instance, for $m=3$, a perfect difference set is 0, 1, 3, 9.) The other lines then are gotten by increasing the subscripts by 1, by 2, etc. It is not too difficult to prove that such a difference set leads to a finite geometry. In fact, the first three properties of a finite geometry and the fifth already hoid. Hence it remains only to show Properties 4, 6, and P7. Here we shall only give an indication of how to show Property 4, leaving the rest to the reader.

Let m^2+m+1 be denoted by s and suppose

$$a_0, a_1, \cdots, a_m$$

is a perfect difference set (mod s). The points P with these subscripts will constitute the line L_0, and the line L_u will consist of the points P whose subscripts are

$$a_0+u,\ a_1+u, \cdots, a_m+u \pmod{s}$$

where u takes the values 0, 1, 2, $\cdots$, m. Let P_i and P_j be two points. We wish to show that they lie on one and only one line L, that is, there is only one set of three numbers: a_k, a_r, u such that $i=a_k+u$ and $j=a_r+u$. Since the a's form a perfect difference set, there will be exactly one pair such that $a_k-a_r=i-j$ (mod s). Then there is exactly one u in the range indicated such that $i=a_k+u$ (mod s). For example, if $m=3$ and the difference set is 0, 1, 3, 9 with $s=13$, suppose we find the line determined by P_5 and P_9. Then $9-0=5-9$ (mod 13) shows that $a_k=9$, $a_r=0$ and $5=9+u$ (mod 13) shows that $u=9$. Hence the points P_5 and P_9 lie on L_9.

Block designs

Recent interest in finite geometries has arisen from their connection with designs used in experiments. Using as an example the finite geometry with seven lines and seven points, suppose one wanted to test seven different varieties of seeds by planting. One might have seven different plots of ground. Let these plots correspond to the lines of the geometry and the varieties of seed correspond to the points. A seed then would be planted in a given plot of ground if the corresponding point occurred on the corresponding line. Then there would be a desirable symmetry of treatment since each seed would be planted in three different plots of ground, each plot of ground would contain three different varieties of seed, and each seed would be competing with each other seed in a plot of ground exactly once. Each finite geometry yields a design with analogous properties.

Desargues' Theorem

One of the most interesting theorems of projective geometry is Desargues' Theorem. We shall show that this theorem holds in every finite geometry which has a co-ordinatization as described above. Recall that this theorem may be stated as follows: If a, b, c and a', b', c' are two sets of three points each such that the lines aa', bb', cc' are concurrent, then the points determined by the following pairs of lines are collinear: ab, $a'b'$; ac, $a'c'$; bc, $b'c'$. To prove this let A, B, C and A', B', C' be triples corresponding to the given points. Then, since aa', bb', cc' are concurrent, numbers r_1, r_2, r_3, s_1, s_2, s_3 can be chosen so that

$$r_1A+s_1A'=r_2B+s_2B'=r_3C+s_3C'.$$

These equations may also be written in the following form:

$$r_2B-r_1A=s_1A'-s_2B',$$
$$r_1A-r_3C=-s_1A'+s_3C',$$
$$r_2B-r_3C=-s_2B'+s_3C'.$$

The left side of the first equation is a triple associated with a point on the line ab and the right-hand side is a triple associated with a point on the line $a'b'$. Since

these triples are equal, each represents a point which is on each of the lines and hence their point of intersection. Similarly the triple on each side of the second equation represents the point of intersection of ac and $a'c'$, while the triples in the third equation represent the point of intersection of bc and $b'c'$. But $r_2B - r_1A + r_1A - r_3C = r_2B - r_3C$ shows the third point lies on the line determined by the first two, that is, the points of intersection of the following pairs of lines are collinear: ab, $a'b'$; ac, $a'c'$; bc, $b'c'$. This is what we wished to prove. The converse of this result may be proved in a similar manner or may be seen to follow from the principle of duality.

The existence of finite geometries

Since finite fields with p^n elements exist for every prime p and every natural number, one may, using the methods above, show that a finite geometry exists having $m+1$ points on a line whenever m is a power of a prime. Furthermore, if the geometry is constructed using the coordinatization above, Desargues' Theorem holds. There are, however, finite geometries with $m+1$ points on a line and m a power of a prime, for which Desargues' Theorem does not hold.

On the other hand, if m is 1 or 2 more than a multiple of 4 and if it contains to an odd power some prime factor which is 1 less than a multiple of 4, there is no finite geometry with $m+1$ points on a line. Thus there is no finite geometry for which $m=6$ since m contains to the first power the prime factor 3. But it is not known whether there is a finite geometry for $m=10$. In fact, the only finite geometries known have m a power of a prime. There is thus a wide gap between the values of m for which finite geometries are known to exist and those values for which it has been proved that no finite geometries exist. The existence of this gap in our knowledge is the reason for much of the interest in this subject today.

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C. E. Rickart, "The Pascal Configuration in Finite Projective Geometry," *The American Mathematical Monthly*, 47 (1940), p. 89.

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Any attempt to be precise and pay attention to meanings would involve us with . . . problems . . . which are really quite irrelevant for mathematics. For mathematics, it is the form that must be considered, and the meaning can be dispensed with.—*Taken from* Logic for Mathematicians *by Rosser.*

The postulates of algebra, and non-Archimedean number systems

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This lecture was given in 1936 at the Galois Institute of Mathematics and is reprinted with the permission of the Galois Institute. Because this article is not readily available to secondary teachers, and because of recent interest in the structure of algebra for high schools, this lucidly written article is recommended for close study.

ELEMENTARY GEOMETRY has traditionally been taught from the logical, postulational point of view. Certain undefined objects and relations are assumed to have certain properties, and the consequences of these assumed properties are then derived by the processes of logic. Elementary algebra courses, on the other hand, frequently appear as a hodgepodge of manipulative processes for solving problems—manipulations with exponents, determinants, etc., and methods of solving equations. The abstract concepts of algebra are gradually built up in the mind of the student by experience without being precisely defined. The manipulative processes are to some extent subsumed under heuristic laws, but few textbooks state these laws in systematic fashion. Perhaps the difference in the treatment of algebra and geometry is largely due to the difference in the times of their development. Elementary geometry was systematized by the ancient Greeks, who had a flair for logical systematization. The fundamental concepts of geometry were ready at hand from experience with physical space. Algebra, on the other hand, was largely developed amid the hustle and bustle of Renaissance and modern times, when the main interest has been in the discovery of new knowledge rather than in systematization.

However, algebra can be rationalized and systematized. This process of rationalization of algebra has two important aspects. In the first place, we may start with the known system of ordinary integers 1, 2, 3, $\cdots$, and their familiar properties, as a basis, and in terms of the integers we may *define* step by step the more complicated and abstract types of numbers, namely, fractions, negative numbers, irrational numbers, complex numbers, etc. These types of numbers were actually introduced into mathematics in this step-by-step fashion, but not on the basis of logical definitions. They were introduced because they were needed in the solution of equations, and were at first regarded as illogical somethings, as is evidenced by the names "irrational" and "imaginary". It was not until the latter half of the nineteenth century that the introduction of all these more complicated types of numbers was placed on a sound and reasonable basis and their properties were derived in a logical way, by Meray, Weierstrass, Cantor, and Dedekind. The process of defining the irrational numbers in terms of the more elementary types of numbers and of proving their properties is a rather involved and logically delicate process. That is a sufficient reason why it is not given in high school and college courses in mathematics.

On the other hand, a postulational basis for algebra is easily understandable, and

from a suitable set of postulates the usual rules of manipulation in algebra are easily derived. If we start out with only a few weak postulates as assumptions, we may find that there is a great variety of instances satisfying the postulates. If, however, these postulates are properly chosen, there may be a surprising number of additional properties which can be defined and discussed, and a surprising number of additional theorems which can be proved. When more postulates are added, there are naturally fewer instances of systems satisfying the postulates, while more and stronger theorems can be proved. This analysis of the relations between various algebraic properties is the second important aspect of the rationalization and systematization of algebra.

In the choice of postulates for elementary geometry, we usually select properties which are closely related to our spatial experience and whose consequences are consistent with our experience. Modifications of such systems of postulates, as for instance the omission or altering of the parallel postulate of Euclidean geometry, may, however, lead to interesting results. In algebra, our choice of postulates is determined by our experience of the elementary properties of addition and multiplication of integers, and by the properties found desirable in connection with the solution of equations and other more complicated problems. We shall start with only a few postulates, namely, the postulates for abstract groups. After several successive adjunctions of new postulates, we shall arrive at a system having all the properties of ordinary algebra with respect to the operations of addition, subtraction, multiplication, and division. Examples will be given at each stage to indicate something of the interest attaching to the various systems.

The undefined objects of an algebriac system may be called numbers or elements. These "numbers" will be denoted by small roman letters, a, b, c, etc. The class of all the "numbers" under consideration will be denoted by $\mathfrak{A}$. There are two operations on numbers, which we shall call addition and multiplication and denote by the ordinary symbols, $+$ and $\times$. However, it is to be understood that the "numbers" for which a, b, c, etc., stand may be any sort of objects, as for example, tables, chairs, chalk, and the operations "$+$, $\times$" may be any sort of operations, subject only to the postulates to be stated.

The postulates for an abstract group:

1. To each pair of numbers a, b, in $\mathfrak{A}$, there corresponds a uniquely determined third number in $\mathfrak{A}$, denoted by $a+b$.
2. The operation $+$ is associative, i.e., $(a+b)+c=a+(b+c)$.
3. There is in $\mathfrak{A}$ an identity element e such that
   (1) $e+a=a$ for every a;
   (2) to each a there corresponds a number $\bar{a}$ such that $\bar{a}+a=e$.

From these assumptions it may easily be shown that the elements e and $\bar{a}$ are uniquely determined, that $a+e=a$, and that $a+\bar{a}=e$. It also follows that for every a and b the equations

$$a+x=b, \qquad y+a=b$$

have unique solutions $x=\bar{a}+b$ and $y=b+\bar{a}$ respectively. The postulates 1 and 3 may be replaced in fact by the statement that if in the equation $a+b=c$, any two whatever of the elements are chosen arbitrarily in the system $\mathfrak{A}$, there always exists in $\mathfrak{A}$ a uniquely determined third element satisfying the equation. The identity element e may be called "zero" in case the operation $+$ is thought of as addition, and it may be called unity if the operation $+$ is thought of as multiplication. In case the commutative law holds:

4. $a+b=b+a$, for every a and b, and the group is called *commutative*, or *Abelian*, after the Norwegian mathematician Abel.

A simple example of an Abelian group is made up of all integers, positive, nega-

tive, and zero, with the operation + as ordinary addition. As an example of a group in which the elements a are not ordinary numbers and the operation + is neither addition nor multiplication, we may consider the group of rotations of a plane about a fixed point 0. Here the elements a are the rotations, and the operation + of combining rotations is simply that of performing them in succession. A rotation is determined by giving an angle θ, and combining two rotations performed in succession corresponds to adding their angles. If we are interested only in the final position of the points of the plane after the rotation, we may suppose that the angle θ lies in the interval $0 \leqq \theta < 2\pi$. If the angle which is the result of adding two or more angles falls outside this interval, multiples of 2π are subtracted to bring it back within the interval. The equation $\theta+\theta=\phi$, where ϕ is a given angle, has in this case two solutions, and the equation $\theta+\theta+\theta=\phi$ has three solutions. This idea is familiar in connection with de Moivre's theorem and the n nth roots of a complex number. Both of the groups just described are Abelian groups. We may readily obtain a noncommutative group by considering all the rigid motions in a plane. The result of performing first a translation and then a rotation is not the same as the result of performing the same rotation first and then the translation.

The postulates for a ring:

1. $\mathfrak{A}$ with the operation + forms an Abelian group.
2. To each pair of numbers a, b in $\mathfrak{A}$ there corresponds a uniquely determined third number in $\mathfrak{A}$, denoted by $a \times b$ or ab.
3. The operation $\times$ is associative, i.e., $(ab)c = a(bc)$.
4. The operation $\times$ is doubly distributive with respect to +, i.e., $a(b+c) = ab+ac$, $(a+b)c = ac+bc$.

From these assumptions it follows that $a \times 0 = 0 = 0 \times a$ for every a, where 0 denotes the identity element for the group $\mathfrak{A}$ with the operation +. A *commutative ring* is one in which the operation $\times$ also obeys the commutative law.

The class of all integers, including zero and the negative integers, with the usual addition and multiplication tables, forms a ring. Another ring is obtained from this one by omitting all the odd integers, since the sum, difference, and product of even integers are always even. This shows that a ring need not contain a unit element for multiplication. As a queer example we may consider the class $\mathfrak{A}$ composed of the four objects "chalk," "red," "chair," "desk." It is necessary to set down the addition and multiplication tables. We suppose them both commutative, and define

chalk + a = a for every a,
red + red = chair,
red + chair = desk,
red + desk = chalk,
chair + chair = chalk,
chair + desk = red,
desk + desk = chair.

chalk $\times$ a = chalk for every a,
red $\times$ a = a for every a,
chair $\times$ chair = chalk,
chair $\times$ desk = chair,
desk $\times$ desk = red.

A mathematically equivalent way of writing down this example, a way which is simpler and more instructive, is to denote the four objects by the symbols 0, 1, 2, 3. Then the addition and multiplication tables are derived from the usual ones simply by subtracting multiples of 4 from the result of the usual addition or multiplication whenever that result is greater than 3. We observe that $1+3=0$, so that 1 and 3 are negatives of each other. Also $2 \times 2 = 0$, so that a product may be zero when neither factor is zero.

An important example of a ring is obtained by taking the class of all polynomials $a_0+a_1\xi+\cdots+a_n\xi^n$ in a single variable ξ. Here the coefficients a_0, a_1, $\cdots$, a_n may be allowed to be complex numbers, or they may be restricted to be real numbers, rational numbers, or even to be inte-

gers. The addition and multiplication of two polynomials are processes performed according to the usual rules. It may be well to notice that the various powers of the variable ξ are merely place indicators which show how to combine the coefficients in forming a sum or product. The variable ξ need not be a number or have any other special properties except the ability to hold a place.

The postulates for a quasi field;

1. $\mathfrak{A}$ contains at least two elements.
2. $\mathfrak{A}$ with the operations $+$ and $\times$ forms a ring.
3. If 0, the identity element for $+$, is omitted from $\mathfrak{A}$, the remaining elements with the operation $\times$ form a group.

Note that for a quasi field we do not assume that multiplication is commutative. Possibly the curious may inquire what would happen in case addition is not assumed to be commutative either. In the postulate systems for groups and rings, the commutative law for the operation $+$ was an independent postulate. For example, any group satisfied all the postulates for a ring except the commutative law for $+$, if we define $a \times b = 0$ for every a and b. When we come to a quasi field, however, we find that the commutative law for addition may be shown to be a consequence of the other postulates, as follows. For any two numbers a and b, we find by using the distributive laws in two orders,

$$(1+a)(1+b) = 1+a+b+ab = 1+b+a+ab,$$

where 1 is the unit element for multiplication. Since subtraction is possible and gives a unique result in a group, we find $a+b=b+a$.

A famous and important example of a quasi field is the system of real quaternions invented by Sir William Rowan Hamilton. Here on account of the non-commutativity of multiplication, the powers of a single letter ξ are not convenient for place indicators. Instead it is customary to use the three letters i, j, and k. A quaternion is then written as $a+bi+cj+dk$, for example, $2-3i+7j+8k$. Addition and multiplication are performed in the usual way, except that the product of any two of the letters i, j, and k is expressed in terms of a single one of them or in terms of 1, the unit for multiplication. The multiplication table may be expressed in the following form:

$$ii = jj = kk = -1, \qquad ij = k = -ji,$$
$$jk = i = -kj, \qquad ki = j = -ik.$$

A *field* is a quasi field in which multiplication is commutative. Important examples of the notion of a field are the class of all rational numbers, the class of all real numbers, and the class of all complex numbers, with addition and multiplication defined as usual. In fact, the postulates for a field and the consequences which may be deduced from them are exactly the familiar laws of algebra in the usual sense for the operations of addition, subtraction, multiplication, and division.

We have noted that the integers 0, 1, 2, 3 form a ring when addition and multiplication are properly defined. We likewise obtain a ring from any finite set of integers 0, 1, 2, $\cdots$, $m-1$, when addition and multiplication are defined as usual except that when the result falls outside the given class, a proper multiple of m is subtracted to bring the result back within the class. Thus for $m=10$, we should have $8 \times 8 = 64-60 = 4$, etc. If m is composite, $m=pq$, according to the ordinary multiplication table we have $pq=0$ in the corresponding ring formed as above, so that the ring cannot be a field when m is composite. On the other hand, it will be a field when m is a prime.

We proceed next to the postulates for a simple order in a field. Any aggregate $\mathfrak{A}$ is said to be simply ordered in case there is a relation $<$ between pairs of its elements, having the properties:

1. For every a and b with $a \neq b$, either $a<b$ or $b<a$ is true.
2. No two of $a=b$, $a<b$, $b<a$ are true simultaneously.

3. The relation $<$ is transitive, i.e., if $a<b$ and $b<c$, then $a<c$.

A given aggregate may be ordered in various ways. For example the positive integers may be ordered as usual, or in the order 1, 3, 5, 7, $\cdots$; 2, 4, 6, 8, $\cdots$.

For the ordering of a field it is desirable to have certain relations between the order and the operations of addition and multiplication. A field may be simply ordered by separating all its numbers except zero into two mutually exclusive classes, called *positive* numbers and *negative* numbers, with the properties:

A. If a and b are both positive, then $a+b$ and ab are both positive.
B. If a is negative, then $-a$ is positive.

Then if $a\neq b$, $b-a\neq 0$ and hence is either positive or negative. If $b-a$ is positive, we say $a<b$. Then the properties A and B imply the properties 1), 2), 3) given earlier. It is interesting to note that the field of all complex numbers may be simply ordered, but not in such a way as to satisfy the conditions A and B. The same statement is true for the finite fields made up of 0, 1, 2, $\cdots$, $p-1$, where p is a prime. To see the truth of these statements for complex numbers, we may define $x+yi<u+vi$ in case $x<u$, or if $x=u$, in case $y<v$. This ordering has properties 1), 2), 3). But an ordering satisfying the conditions A and B must have 1 positive and -1 negative. If i were positive, then $ii=-1$ would be positive, and if i were negative $-i$ would be positive and hence $(-i)(-i)=-1$ would be positive. Thus the required ordering of all complex numbers is impossible.

Let us consider now only orderings of fields satisfying the postulates A and B. Since 1 is positive, all the numbers formed from 1 by addition, multiplication, and division must be positive also. We obtain in this way the class of positive rational numbers. We easily find that if $a<b$, then $a+a<a+b<b+b$, or $a<(a+b)/2<b$, so that between every two numbers of an ordered field there lies a third, i.e., every ordered field is *dense*.

The Archimedean principle is used both in geometry and in algebra. In geometry this principle states that if two line segments AA' and BB' are given, the first may be applied to the second as a measuring stick beginning with A at B, and after a finite number n of applications A' will reach beyond B'. If AA' and BB' denote also the magnitudes of these lines, we may say that there is an integer n such that $n\cdot AA'>BB'$. In an ordered field the Archimedean principle states that if a and b are arbitrary positive numbers there is an integer n such that $b<na=a+a+\cdots+a$ to n addends. This principle holds in the field of all real numbers, and hence in all subfields of the field of real numbers. The realization of this fact eventually banished from the calculus the use of infinitesimals as they were described and apparently conceived by the forerunners of Newton and Leibnitz and by some of their successors also. By that I mean the concept of an infinitesimal as a number not zero, but such that an infinite number of repetitions of such a number must be added together to equal or exceed a given finite number. It is true that these early workers more often than not obtained correct results by these erroneous methods, but that was because their actual calculations were guided by correct intuitive perceptions of the situations they were studying.

Now there are no Archimedean ordered fields essentially different in their mathematical properties from the subfields of the real numbers. However, it is not so obvious that there exist non-Archimedean ordered fields, since many fields cannot be ordered. We will now proceed to study an example.

We may start from any ordered field $\mathfrak{A}$, for example, the field of all real numbers. These numbers will be denoted by small roman letters with subscripts. The numbers to be defined will be denoted by the corresponding letters without subscripts.

That is, we shall set

$$a=\sum_{i=k}^{+\infty} a_i\xi^i, \qquad b=\sum_{i=m}^{+\infty} b_i\xi^i, \qquad c=\sum_{i=n}^{+\infty} c_i\xi^i$$

where nothing is to be said about convergence, and ξ is not supposed to stand for a number of $\mathfrak{A}$, nor indeed for anything in particular. The various powers of ξ merely serve as convenient place indicators in carrying out the processes of addition and multiplication yet to be defined. This is an extension of the remark made already in connection with the ring of all polynomials in one variable. The first terms of the formal power series for a, b, c may have for their indices k, m, n respectively, any integer, positive, negative, or zero. It is understood that for every such number a, the first coefficient indicated, namely, a_k, is not zero, unless all the a_i are zero. Then we define addition and multiplication of the elements a, b, etc., to be formally the same as for ordinary power series. That is, if $c=a+b$, $c_i=a_i+b_i$, and $n=$lesser of k and m. If $c=ab$, $c_i=\sum_{s+j=i} a_s b_j$, and $n=k+m$. In the formula for the general term c_i of the product ab, there are only a finite number of nonzero terms, and hence there is no question of convergence.

It is easy to verify that the class of all such elements a, where the coefficients a_i range over a field $\mathfrak{A}$, and where $+$ and $\times$ are defined as stated, forms a field $\boldsymbol{A}$. The zero of the field $\boldsymbol{A}$ has zeros for all its coefficients. The negative $-a$ of a is obtained by changing each coefficient a_i to its negative. The unit of the field $\boldsymbol{A}$ has the unit 1 of the field $\mathfrak{A}$ as coefficient of ξ^0, and zeros of $\mathfrak{A}$ for the other coefficients. We shall verify that every element a of $\boldsymbol{A}$ has a reciprocal b. For this to be true, it is sufficient that the equations

$$m=-k,\ a_k b_m=1, \quad a_k b_{m+1}+a_{k+1}b_m=0,$$
$$a_k b_{m+2}+a_{k+1}b_{m+1}+a_{k+2}b_m=0,$$
$$\cdots\cdots\cdots$$

be satisfied by properly chosen b_m, b_{m+1}, $\cdots$. Since by hypothesis $a_k\neq 0$, the coefficients b_m, b_{m+1}, $\cdots$ may be determined successively by solving one equation at a time. For example, if $a=1-\xi$, then $b=1/a=1+\xi+\xi^2+\cdots$. If $a=\xi^{-1}-1$, then $1/a=\xi+\xi^2+\xi^3+\cdots$.

Since the original field $\mathfrak{A}$ is ordered, it is easily seen that the constructed field $\boldsymbol{A}$ may be ordered by classing a number a as positive if the first nonzero coefficient a_k is positive. The properties A and B are obviously satisfied. Then we see at once that

$$1>n\xi=\xi+\cdots+\xi \quad \text{to } n \text{ terms},$$
$$\xi>n\xi^2,$$
$$\xi^{-1}>n, \text{ etc.},$$

no matter how large the integer n is. Thus every element a for which the index k of the first nonzero coefficient a_k is positive is an infinitesimal with respect to unity in the original sense of the word. These infinitesimals exist only in a strange number world, different from the one we ordinarily use, but nevertheless subject to all the ordinary laws of algebra, except for properties related to the Archimedean property.

I hold every man a debtor to his profession, from which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto.—*Bacon.*

Some foundations of Soviet educational power[1]

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As teachers we cannot learn too much about the purposes and structure of Soviet education: " . . . our generation must build an educational structure adequate for all needs . . . ."

I

ONE of the most important problems now facing the American people is this: "What shall we do in the face of the Soviet challenge in education?"

We are increasingly aware that Soviet education has become a principal instrument of Soviet policy, that education in the U.S.S.R. is a tool with which the government of the country is building an immensely powerful and productive society. There is a growing feeling that the safety of our children, and of their children, depends on the readiness of our generation of Americans to make the necessary sacrifices to equip young Americans for survival in a world where they may be greatly outnumbered by well-trained communists.

Today I am concerned chiefly with this aspect of the question: How did the Soviet Union achieve its present position in education? If we are to act intelligently ourselves, we should first understand how the people of the Soviet Union have made use of education as an instrument for the achievement of their national objectives.

Not enough is known about this subject. We need much research in this aspect of Soviet education. In the meantime, on the basis of my limited observations in the Soviet Union, I would like to offer the following comments concerning the foundations of Soviet educational power.

[1] A speech presented before the Educational Press Association of America, NEA Convention, Cleveland, Ohio, July 1, 1958.

II

First, I must say that the Soviet process of education would not have been possible under our form of government. The last thing we should do is to try to imitate Soviet education or any aspect of Soviet educational policy. We must meet the challenge on our own terms, in harmony with our own traditions. I am convinced that our educational foundation is strong enough to meet any competition. Once we clearly understand the nature of the Soviet challenge we can meet it.

I believe Soviet educational achievement is built on four cornerstones. The first is a system of control, which has made it possible to achieve a degree of universality and high academic standards, over an area nearly three times that of the United States and a population a third again the size of the population of this country.

The second cornerstone of Soviet education is an educational philosophy that must be understood before Americans talk about imitating Soviet educational achievement.

A third source of Soviet educational strength is the Soviet system of financing education, which has resulted in major

educational advancements in a relatively short period of time.

Finally, the union of educational workers had much to do with the development of Soviet education.

III

As a first step toward such understanding, we must know the power structure of Soviet education.

The Communist Party is in complete control of education in the Soviet Union. The real leaders in education in the U.S.S.R. today have come through a lifetime of communist indoctrination. This communist training starts in the nursery schools, where between the ages of 3 and 7 the children learn such communist virtues as obedience, loyalty to the group, industry, and self-control. After they go to school they usually join the pioneer organizations, and here again they learn to be good communists while at the same time having fun developing hobbies and following elective courses not permitted in the regular curriculum. In the late teens, a chosen few, the pick of the youth of the Soviet Union, join the Komsomols. They stay in this organization until they reach the middle twenties. Then those who make the grade may be admitted to the Communist Party. In the meantime, they have gone through their educative process and become specialists in the many fields of learning which are needed in the building of the communist state.

The party membership permeates the entire educational structure of the U.S.S.R. This structure, starting from the top, is as follows:

1. *The academies of sciences.* Each of the Republics which compose the U.S.S.R. has an academy of sciences and an academy of pedagogical sciences, and the membership in these academies is the ultimate goal of the Soviet professional man or woman. I spent some time in the Academy of Pedagogical Sciences of the Russian Republic, which embraces 80 per cent of the land area of the entire Soviet Union. The Director of the Institute is a man named Kairov. He has a staff divided into eight institutes. His specialists are responsible for developing ways to put communist educational theory into practice.

   The Academy develops new ideas, and evaluates the experiences reported to it by educators throughout the Union. It is kind of an educational quarterback which tells the rest of the educators what to do and how to do it.

2. Executive responsibility for education in the U.S.S.R. is divided between a *Ministry of Higher Education*, which is responsible for certain types of higher educational institutions in all of the Republics of the Union, and *Ministries of Education*, which exist in each of the Republics, and which control elementary and secondary education, teacher education, and certain types of special schools in each of the Republics. Because the Russian Republic is so much larger than all of the others, and because of its special political relationships within the Union, it is my impression that the Ministry of Education in Moscow is, in fact, the Ministry of Education for the whole of the U.S.S.R. Moscow makes every effort to play up the importance and the independence of the peoples of the other Republics. However, in international relationships the representatives of the Ministry in Moscow appear to set policies for the entire U.S.S.R.

3. Each of the higher educational institutions under the supervision of the Ministry of Higher Education has an *academic council* which has much to say concerning what goes on within the institution. The actual degree of independence of these councils has varied from time to time; during "liberal" periods in the U.S.S.R. these councils have increased responsibilities. When "Stalinist" tendencies are dominant,

the responsibility of the academic councils also may be diminished.

4. Schools in cities and other political units are controlled by *school departments*, under the councils or Soviets, which correspond in many ways to American school boards. This is one of the many areas where research is needed to discover how Soviet education is both like and unlike American education. In a city such as Leningrad, where there are 21 school districts, there is a supreme school department which exercises executive responsibility for the operation of schools throughout the city, and also for a variety of services related to the school. The same is true of the city of Tashkent in Central Asia. In this instance, the president of the school department, Kaidirov, also is Superintendent of Schools, and likewise is president of the political Soviet which governs the Tashkent area.

5. Finally, there are *parents' committees* that exercise considerable authority over nonacademic matters such as the building and the equipping of schools, and the providing of necessary services. This appears to be particularly true in rural areas. Many schools are being built on collective farms out of the profits made by these farms. The farmers seem to act in a group, with some of the responsibilities of the American PTA's and also of American school boards. They decide what kind of buildings to build, in co-operation with the supervising educational departments; they appropriate the money out of their earnings; they supervise the construction; and when the building is finished, they take a great deal of interest in what goes on inside it.

I had the impression that education in the Soviet Union is a matter of profound concern to people at the grass roots. The success of the Soviet educational system is firmly grounded in this grass-roots support. However, policies are dictated from the highest level, and the grass-roots leaders implement these policies.

IV

Another cornerstone of Soviet educational power is Soviet educational philosophy, which is founded on the teachings of Marx, Lenin, and other Soviet leaders, and on ideas borrowed from many countries.

There was a time in the 1920's when Soviet educators borrowed heavily from American philosophers, especially Dewey. They adopted many features of the Winnetka and Dalton plans. Later, Soviet inspiration came from more traditional European sources. Still later, they began to re-examine their own past; education then became more nationalistic, and more traditional.

Thus, as Kairov told me, Soviet education today is a unique structure founded on elements borrowed from other cultures, on communist philosophy, on nationalism, and on faith in science. It is pragmatic; it insists on uniformity as a foundation of democracy. It is like a column of well-disciplined troops marching toward an unknown goal. Like marching troops, Soviet education is capable of a right-about-face in response to orders.

Certain fundamental ideas permeate Soviet education. In a conversation with authorities in the Moscow Ministry of Education, I was told that their basic concept of the nature and function of education could be summed up this way:

> The minds of our children are the greatest natural resource of our nation.
>
> The function of education is to help each child to achieve his maximum potential contribution to his nation.

This is the reason why universal education has become a principal political goal in the U.S.S.R.

Another Soviet idea is expressed in this saying: "Children may be born healthy,

but they cannot be born educated."

Therefore the educative process is a principal concern of the Soviet people. It is assumed that the child should not be expected to be responsible for his own education, that this is an imperative responsibility of society as a whole.

Yet another popular Soviet idea is this: All normal children are equal, and capable of doing equally well in all studies.

This means that all children in the Soviet Union, including those in the ballet schools, are required to learn the same common core of knowledge, which includes large doses of mathematics and science. This means that the Soviet Union does not believe in individual differences and does not make provision for these differences in the elementary and secondary schools.

To my way of thinking, this is one of the most serious weaknesses of Soviet education today. When I protested that every individual is unique, with his own strengths and weaknesses, I was told by a Deputy Minister of Education, "Your concern for individual differences is a fraud. It is nothing but class education." It was impossible for me to convince her that we have genuine concern for the individual, that we try to discover individual needs as well as individual competencies, and that we try to "temper the wind to the shorn lamb."

Another Soviet educational idea is expressed in what is almost a proverb: "There are no bad children, but there are bad teachers."

This means that if a normal Soviet child fails in school it is not the fault of the child, but that of the teacher, or his parents, or something in his environment.

Subnormal children are isolated in special schools, under teachers who are specialists in defectology.

While teachers work short hours in the Soviet Union as compared to the work load of teachers in many other countries, there is a terrific pressure on them to deliver. If their children fail it is a reflection on them. In one school I visited with more than 1,100 children in it, last year only 14 children failed to be advanced with their classes. In another, a boarding school with 180 boys and girls, no child failed to advance with his class and there was a total of only 8 failing grades.

It seems to be a reasonable assumption that some children are passing who might not pass under a different system.

The effect of Soviet education philosophy on both teachers and pupils is profound. The children are expected to work long and hard. The work norms set up for Soviet children have, according to the Ministry of Health, in the past threatened the health of the Soviet children. Today there are doctors and nurses in Soviet schools who watch the children, and who are responsible for protecting them from overwork. Those who are convalescent are sent in the summertime to Forest Schools, where they have a chance to catch up. Children who are invalids go in the summer to Sanitarium Schools, where they receive special training suited to their physical and mental capacities.

It would be wrong to assume that authorities in the Soviet Union are unfeeling in their attitude toward education of their children. As a matter of fact, children appear to be a pampered class. Their education is motivated not only by communist theory, but by real love, by a desire to help these children to achieve a better life for themselves than their parents and teachers have ever known. Kairov, President of the Academy of Pedagogical Sciences, put it in this way: "These children are our future."

V

An essential cornerstone of any educational system is money.

Where does the money come from for education in the Soviet Union? How much money is the U.S.S.R. spending for education as compared to what we spend in the United States? No accurate answers to these questions are available.

The national budget of the U.S.S.R. has a large amount earmarked for education at all levels. This money is distributed through the administrative channels of the Union and of the constituent Republics. This fund includes money earmarked for television and radio. Thus it is impossible for us to ascertain what proportion of the national budget is going into what we would call education. However, we should remember that the radio and television are regarded in the Soviet Union as teaching aids whereby both adults and children are to be educated.

Each Republic in the Union has its own budget, and here substantial proportions are earmarked for education. I tried to find out what part of the budgets in the Russian Republic and the Uzbek Republic was thus earmarked. I could get no firm answers, but the people with whom I was talking maintained that more than 50 per cent of the budget of each Republic is reserved for education.

Local Soviets also appropriate money for education. These organizations correspond roughly to the governing bodies in our cities and counties. The operation of schools under their jurisdiction is their responsibility, under edicts issued from Moscow and from the capitals of their own Republics. They appropriate substantial sums of money to supplement the money that comes from Moscow and from their own capitals.

Yet another source of funds for Soviet education is the noneducational Ministries, both of the national government and of the various Republics. The Ministries, of Defense, for example, operate special schools both for the children of people who work in military establishments, and for military personnel.

Finally, industry contributes heavily to the support of education in the U.S.S.R. The great textile industry, for example, builds schools and turns them over to the local ministries of education and the local school Soviets for operation.

The total contribution from industry and from various ministries to education in the U.S.S.R. is substantial. I tried to find out what proportion of the gross national production of the U.S.S.R. goes to education. I was told that their system of bookkeeping makes it impossible to isolate expenditures for education. However, it was the consensus that nearly 13 per cent of the GNP is now going into education. I do not vouch for this figure.

A powerful factor in Soviet education is the U.S.S.R. Trade Union of Educational Workers. This Union boasts about four million members, and includes all whose work is connected with education. It includes charwomen, school administrators, teachers, and university presidents.

It appears to be a buffer between the individual and the Soviet state, and a means for collective action on national policies in education. The Union offers the teacher a kind of protection from a variety of pressures. If unfounded charges are brought against a teacher, the Union will normally help him. If a supervisor does not like a teacher, the latter may look to the Union for protection.

The Union helps in other ways. It provides a credit bureau which assists teachers financially; for example, home ownership in Uzbekistan is actively supported by the Union. I was told that a teacher who wants to own his home in this Republic applies to his Union for a loan, and may receive 10,000 rubles, which may be repaid over a period of 20 years with a low rate of interest. Furthermore, the Union may give the teacher certain building supplies sufficient for a simple structure. Then the teacher gets from the local political Soviets a grant of land, which is his to have and to hold. This land is tax-free for the first ten years. The teacher may will this property to his children.

The Union maintains welfare activities for its members. It operates sanitariums and rest houses. When teachers travel abroad, the Union may pay the cost.

VI

Education in the U.S.S.R. is the business of all of the people. It is an effective

political weapon because the people believe in it and support it. Through education, the peasant believes his son may one day become a great scientist, or head an expedition to the moon, or become an important political figure.

Education is the chosen instrument whereby the leaders of the Soviet Union expect to build a great nation of irresistible power. If we expect to be able to compete, we must renew our faith in American education; we must show that we, also, are willing to sacrifice, to enable our children, and our children's children, and their friends around the world, to survive as free men and women.

We must strengthen the foundations of American education. On this foundation our generation must build an educational structure adequate for all needs, strong enough to stand against any storm that may arise.

A Parent's Plea

My little boy is eight years old,
He goes to school each day,
He doesn't mind the tasks they set,
They seem to him but play.
He heads his class at raffia work,
And also takes the lead
At making dinky paper boats—
But I wish that he could read.

They teach him physiology,
And oh, it chills our hearts
To hear our prattling innocent
Mix up his inward parts.
He also learns astronomy
And names the stars by night,
Of course he's very up-to-date—
But I wish that he could write.

They teach him things botanical,
They teach him how to draw,
He babbles of mythology,
And gravitation's law;
And the discoveries of science
With him are quite a fad,
They tell me he's a clever boy—
But I wish that he could add.

—*Peter McArthur*, 1907

Mathematics instruction via television

G. DON ALKIRE, *Fresno State College, Fresno, California.*

To meet the needs of increased enrollments and to provide for in-service training, institutions are turning to courses offered via television. This report illustrates one technique used to evaluate such courses.

Statement of the problem

THE PURPOSE of this article is to report the results of an investigation designed to determine the effectiveness of instruction via television when used in a certain course in mathematics. Results obtained by a television class are compared with results secured in the same course presented in three other types of classes.

In recent months this investigator has taught essentially the same course to four different types of classes: (a) a television class, (b) a summer session class, (c) an extension class, and (d) a regular session class. For each class, the same basic methodology of instruction was employed, students were given the same written lessons, midterm, and final examinations, and the same keys were used in scoring papers.

The television course

In the fall semester of 1956–57, Fresno State College offered the course Advanced Problems in the Teaching of Arithmetic by television. Station KMJ-TV graciously extended its services for the purpose. Between October and February, fifteen live programs were telecast. Each was an hour long.

Students who enrolled for two units' credit were required to appear at the college for a midterm and a final examination. In addition, each student submitted fifteen lessons during the progress of the course. These lessons were printed in the appendix of a syllabus composed by the instructor. The syllabus was designed to orient the student for each telecast.

Organized forms of solutions of certain important types of problems were furnished. The merits of forms and algorisms which were considered acceptable, in that they were consistent with the underlying philosophy adopted in the course, were presented. Alternative forms which did not meet this criterion were supplied, along with arguments designed to demonstrate wherein they could not be defended as being teachable.

The course was designed for classroom teachers to fulfill a requirement for the elementary teaching credential. A partial list of topics covered in the syllabus and discussed in the telecasts follows:

1. Finding the whole quantity when a part is known.
2. Fractions, per cents, decimals.
3. Cost-overhead-profit; problems involving per cents.
4. Social, logical, and psychological criteria for developing an adequate arithmetic program.
5. Developing generalizations; discovering rules.
6. Decimal nature of our number system.
7. Positional number systems: base 4, 6, 10, and 12.
8. Interest; sixty-day method.
9. Algorisms; short cuts; rounding numbers.
10. The square of the sum of two numbers.
11. Square root; use of the square of $(10t+u)$.
12. Concept of pi; deriving formulas; physical applications.
13. Deriving formulas for the arithmetic progression.
14. A democratic principle and its implications for education, especially as it relates to methodology of instruction.
15. The uniqueness principle.

THE AUTHOR wishes to express appreciation to the Director of TV Projects at Fresno State College for guidance in the execution of the course and to the TV Evaluating Committee for the use of the material contained in its report.

Very few of the problems in the lessons were merely examples designed to test accuracy of performance of the fundamental operations; most of the problems were applications of the theory presented and were designed with the anticipation that most students might find them provocative. In addition to the problems in the syllabus, an effort was made to furnish one problem near the end of each telecast to test the originality and resourcefulness of the student. This procedure had two additional purposes: (1) to check whether students attended the telecasts, and (2) to entertain the many unregistered viewers.

Although the syllabus, aside from the telecasts, served as the principal source of information for the course, the students were encouraged to study certain topics covered in the textbook, *Teaching Children Arithmetic*, by R. L. Morton. One-fourth of the midterm examination was devoted to testing items covered in the textbook assignments, and although readings from the textbook were not discussed in the telecasts, the class obtained an average of about 70% on this part of the examination.

Methodology of instruction

It is supposed that society desires the student of mathematics to possess ability to reason his way through new problematic situations involving quantity and to fashion adequate methods and procedures for communication of mathematical information. Further, it is supposed that learning constitutes development or modification of insight.

For years, this instructor has adopted a reflection-level type of teaching and learning—one in which the student gets experience in meeting situations in which there is a conflict of ideas; where various hypotheses as to possible methods of attack are advanced and entertained and tested; where meanings are discovered by the learner, formulated by him, and then generalized by him in such a form that he sees he can apply them.

Emphasis has been given to the development of rules by the learner in place of the telling of rules by the teacher. Meanings have not been imparted directly from teacher to pupil for memorization in set language patterns. Acceptance of glib verbalism as evidence of sound learning has been avoided. Meanings have been considered as relative to the experience of the learner. Experiences to get meanings must be ordered, that is, they must not come by chance but by design. It takes a long period of systematic training for most students to attain functional competence in arithmetic. To help promote the desired functional aspects of learning, this instructor has taught in terms of mastery of key concepts and fundamental ideas, kept transfer of training in mind, and endeavored to encourage reflective thinking —scientific thinking. On the part of the student, personal conviction is genuine conviction, and it is likely to attend a serious educational experience.

Although this instructor has taught arithmetic for reflective thinking in the classroom, he does not claim to have done so during the telecasts, with no students present in person. He is, however, in possession of considerable evidence which would seem to indicate that reflective thinking obtained to a higher degree than he had anticipated. Further, he feels justified in claiming that he taught the students how to teach reflectively.

As for the mechanical aspects of the television presentations, the studio employed two television cameras. On each of four large easels was mounted a large pad of paper. The instructor used heavy black grease crayons to demonstrate algorisms and organized solutions for the problems studied. He wore a microphone, attached to a long extension cable, which facilitated movement to any easel. He could raise a provocative arithmetical question, entertain several methods of solution, write them down at the various easels, and encourage his viewers to consider the shortcomings of certain methods and the merits

of others. On one program in which formulas for the arithmetic progression were developed, the grease crayons were used to write on one sheet of paper some twenty feet long. On three or four occasions, actual objects, such as cones, cylinders, and spheres, were employed to aid in the development of formulas. Several dinner plates of different diameters were measured in an effort to gain the notion that the ratio of the circumference of any circle to its diameter is a universal constant. Such extra visual aids were employed very sparingly during the telecasts.

The questionnaire

The college appointed a committee to evaluate the television course. The committee constructed a questionnaire sufficiently general in nature as to be employed to aid in evaluating other television courses. This questionnaire was administered twice to the students taking the course, once at the midterm examination and again at the final examination. The directions printed at the top of the questionnaire urged students to "make sure you select the response that most nearly coincides with your judgment."

A summary of sixteen of the items in the questionnaire is presented below in Table 1. Each question had five possible answers, except the tenth question, which had six. These answers were assigned weights as shown in the table. Approximately 80 students responded to each question. The symbols used in the last four columns of the table have the following meanings:

X = responses obtained after seven television lectures,
Y = responses obtained after fifteen television lectures,
$\overline{X}$ = the mean of the weighted X's on a question,
$\overline{Y}$ = the mean of the weighted Y's on a question.

For each question, except the sixth, the phrase in italics represents both the most frequent response and the average weighted response.

TABLE 1

Summary of Student Response on Evaluation Questionnaire Pertaining to the Course, "Advanced Problems in Teaching Arithmetic," Presented via Television

Question	*Weight*	X	Y	$\overline{X}$	$\overline{Y}$
1. Compared with regular classroom lectures, I thought the TV lectures were:	5 much more interesting	15	16		
	4 *more interesting*	37	33		
	3 about the same	29	33		
	2 less interesting	2	1		
	1 much less interesting	0	0		
		83	83	4.30	3.77
2. Compared with classroom lectures, the TV lectures made understanding the material:	1 much more difficult	1	0		
	2 more difficult	6	7		
	3 *about the same*	48	42		
	4 more lasting	21	21		
	5 much more lasting	6	10		
		82	80	3.26	3.42
3. Compared to regular classroom lectures, I found my retention of the material:	1 much less lasting	3	2		
	2 less lasting	8	6		
	3 *about the same*	36	39		
	4 more lasting	29	24		
	5 much more lasting	5	10		
		81	81	3.31	3.42

Question	*Weight*	*Y*	*Y*	$\bar{X}$	$\bar{Y}$
4. Compared with what I have learned from classroom lectures, I am learning by TV:	5 much more	11	17		
	4 somewhat more	28	21		
	3 *about the same amount*	36	35		
	2 somewhat less	6	7		
	1 much less	0	0		
		81	80	3.54	3.12
5. As compared with those from regular lectures, my notes are:	1 much poorer	3	3		
	2 poorer	9	8		
	3 *about same quality*	34	24		
	4 better	22	26		
	5 much better	14	18		
		82	79	3.30	3.61
6. In my judgment, the elimination of class questions made learning:	5 much more efficient	3	9		
	4 more efficient	24	25		
	3 *about the same*	23	20		
	2 less efficient	29	23		
	1 much less efficient	1	4		
		80	81	2.99	3.15
7. Compared to regular classroom lectures, I was able to give the TV lectures:	5 much more attention	20	10		
	4 *more attention*	40	35		
	3 about same attention	18	21		
	2 less attention	2	16		
	1 much less attention	1	0		
		81	82	3.94	3.47
8. In general, I believe the TV lectures were delivered:	1.5 much too rapidly	0	0		
	3.5 too rapidly	1	2		
	5.0 *about right speed*	72	75		
	3.5 too slowly	7	4		
	1.5 much too slowly	0	0		
		80	81	4.82	4.89
9. I believe the TV lectures contained:	1.5 far too much material	0	1		
	3.5 too much material	4	4		
	5.0 *about right am't material*	69	66		
	3.5 too little material	8	10		
	1.5 much too little material	0	0		
		81	81	4.78	4.69
10. Indicate the extent to which the material covered is appropriate to your needs:	5.0 very appropriate	23	34		
	3.5 *mostly appropriate*	33	24		
	2.5 only slightly appropriate	11	4		
	1.0 inappropriate	3	2		
	1.0 too high a level	9	12		
	1.0 too low a level	0	1		
		79	77	3.42	3.62
11. As compared with other serious TV presentations, these lectures were:	5 superior	20	27		
	4 *somewhat better than average*	33	29		
	3 about average	19	20		
	2 somewhat below average	2	0		
	1 definitely inferior	0	0		
		74	76	3.96	4.09

Question	*Weight*	*X*	*Y*	*X*	*Y*
12. The quality of picture reception was:	5 *very good*	44	56		
	4 good	29	27		
	3 fair	8	6		
	2 bad	0	0		
	1 very bad	0	0		
		81	89	4.44	4.56
13. The quality of sound reception was:	5 *very good*	50	56		
	4 good	29	23		
	3 fair	3	2		
	2 bad	0	0		
	1 very bad	0	0		
		82	81	4.57	4.67
14. Generally, the visual aids were:	5 *very easy to see*	30	35		
	4 easy to see	32	26		
	3 adequate	14	15		
	2 fair	4	4		
	1 poor	1	1		
		81	81	4.06	4.11
15. As helps in understanding the material, the visual aids were:	5 superior	17	22		
	4 *very good*	44	47		
	3 adequate	20	11		
	2 fair	1	2		
	1 poor	0	0		
		82	82	3.70	4.09
16. The visual aids were:	1.5 far too numerous	0	0		
	3.5 too numerous	0	1		
	5.0 *about right*	75	79		
	3.5 too few	7	2		
	1.5 far too few	0	0		
		82	82	4.88	4.94

From the responses to Questions 1, 8, 9, and 10, the students felt that the television lectures:

(a) were interesting enough,
(b) were delivered at a satisfactory speed,
(c) contained the right amount of material,
(d) covered appropriate material.

From the responses to Questions 2, 3, and 4, the students felt that the television lectures compared favorably with regular classroom lectures on the following characteristics:

(a) understanding of the material,
(b) retention of the material,
(c) amount of learning.

During the execution of the course, the instructor was interested in knowing whether the elimination of class questions had a significant deterring effect on learning. If it did, the distributions of responses to the sixth question of Table 1 do not reveal it. Out of the total of 161 responses, 104 indicated that learning efficiency was not lowered due to the elimination of class discussions, and 57 indicated that it was.

There were 85 students who took the course. Of these, 71 indicated that they were teachers, 47 of these holding provisional teaching credentials and 24 possessing regular teaching credentials. Out of

64 who responded to a question on classification of teaching position: 22 taught in the primary grades, 22 taught in the intermediate grades, 5 taught in the upper elementary grades, 11 taught in the junior high school, and 4 taught in the high school.

Evaluation of achievement

The midterm examination was divided into two parts. The first part pertained to content covered in the syllabus and presented in the telecasts; the second part pertained to theoretical topics from the syllabus and text. On the first part, the average was 74.6 with a standard deviation of 15; on the second part, the corresponding measures were 70.4 and 12. The coefficient of correlation between the two distributions was .60.

On the final examination, the average was 80.4 with a standard deviation of 14. Table 2, below, exhibits frequencies of scores on the midterm examination, the final examination, and the course grade.

TABLE 2

Distribution of Mathematics Scores for the Television Class

Scores	*Midterm examination*	*Final examination*	*Course grade*
90–100	9	26	12
80– 89	15	22	32
60– 79	46	31	31
40– 59	13	5	9
0– 39	2	1	1

Table 3 exhibits frequency distributions of course grades obtained by four different types of classes that have, in recent months, completed the course under consideration in this article. The same instructor taught all four classes.

It would not seem unreasonable to conclude from the data in Table 3 that the television class achieved to a significantly higher degree in the mathematics course than did the summer session class, the extension class, or the regular session class.

TABLE 3

Distribution of Course Grades in Mathematics for Four Types of Classes

Range of scores	*Television*	*Summer session*	*Extension*	*Regular*
90–100	12	3	4	1
80– 89	32	7	6	12
60– 79	31	14	11	17
40– 59	9	6	9	8
0– 39	1	3	1	1

It is interesting to note the size of the total television audience drawn to the mathematics program. According to a survey conducted on January 5, 1957, approximately 5,500 homes were tuned in to KMJ-TV Channel 24 during the hour 9:00–10:00 A.M. This means that somewhat more than 11,000 people watched the mathematics program. This figure is interesting because at the same hour the other two local channels were carrying a message to the nation delivered by the President of the United States.

Summary

The questionnaire evaluated three aspects of the television course: (1) the degree of instructiveness, (2) the quality of the instruction, and (3) the adequacy of the technique of televising.

1. Instructiveness: Compared with regular classroom instruction, the television instruction
   a. was more interesting;
   b. made understanding a little less difficult;
   c. made retention a little more lasting;
   d. made for learning of a little more material;
   e. made for a little better quality of student notes;
   f. made learning as efficient despite the complete elimination of class questions and discussions;
   g. commanded better attention.
2. Quality of Instruction: The television lectures
   a. were delivered at about the right speed;
   b. contained about the right amount of material;
   c. contained material which was mostly appropriate to the needs of the student;
   d. were somewhat better than other serious TV presentations.

3. Technique of Televising: According to the TV class
   a. the quality of picture reception was very good;
   b. the quality of sound reception was very good;
   c. the visual aids were very easy to see;
   d. as aids in understanding, the visual aids were very good;
   e. the visual aids were numerous enough.

Based upon achievement scores, it would seem that instruction of the mathematics course via television was as effective as regular classroom instruction of the course in three other types of classes, namely: a summer session class; an extension class; and a regular session class.

Remarks and recommendations

The television instructor of mathematics who adopts a methodology based upon sound psychological principles and acquaints himself with the mathematical habits and mathematical personalities of his students can, through carefully prepared syllabi and planned assignments conscientiously checked for all students, teach by television much as though his students were present in a classroom.

It is hoped that the administrator of educational television, as well as the television instructor of mathematics, will find the present investigation of some value.

What's new?

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Complex figures in geometry

JOHN D. WISEMAN, JR., *Tenafly High School, Tenafly, New Jersey.*

Here is more "food" for your gifted students.

ONE OF THE MAJOR DIFFICULTIES faced by high school geometry students is the analysis of complex figures. Complex figures may be defined as figures consisting of "overlapping" triangles, angles, or straight line segments. A student usually has no method, other than trial and error, for analyzing such figures.

The following method has been tried with high school geometry students and seems appropriate for use with better students. This method is suggested by a chapter in *Geometry and the Imagination* by Hilbert and Cohn-Vossen.[1] This chapter considers only a special set of figures called "plane configurations." A plane configuration is defined as a set of points and straight lines in a plane such that every point is on a certain number of lines, and every line is on a certain number of points. A plane configuration is described by the following symbol: $(\rho_\gamma l_\pi)$. The definition requires that the four variables have values such that $\rho \cdot \gamma = l \cdot \pi$. A triangle is a plane configuration which has the symbol $(3_2\ 3_2)$. This symbol has the following meaning. Three points of intersection are each on two straight lines, and three straight lines are each on two points of intersection.

This idea may be extended to describe the incidence relations of figures in high school geometry. For example, Figure 1 is described by the following equation.

$$1_3+3_2=1_3+3_2$$

[1] D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination* (New York: Chelsea Publishing Co., 1952), p. 95 ff.

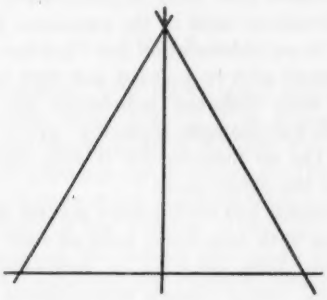


Figure 1

The left-hand side of the equation means that one straight line is on three points of intersection, and the other three lines are each on two points of intersection. The right-hand side of the equation means that one point of intersection is on three straight lines, and the other three points of intersection are each on two lines.

If each number is multiplied by its subscript, then the following equation is obtained: $3+6=3+6$, or $9=9$. This number indicates the degree of complexity of the figure, that is, the larger the number the more complex the figure. Note that the same number is obtained by different procedures on each side of the equation. The reason why this occurs is explained by the fact that whether one counts the number of times that points are incident with lines or the number of times that lines are incident with points one gets the same total.

A good beginning for high school students is to ask them to draw all the figures which have four straight lines. Then the students can write the equation for each

of these figures. However, one agreement is necessary first. The lines must be written on one side of the equation, and the points of intersection on the other. Perhaps the agreement easiest to remember is this: write the *l*ines on the *l*eft. Since it is necessary to reach this agreement, the equation can properly be called the "code equation" of the figure. The number which is obtained by multiplication on both sides can be called the "code number."

Students also need to understand the abbreviations used in the equation. Thus "3_2" is an abbreviation for "$1_2+1_2+1_2$." It should also be pointed out that numbers with different subscripts are not added. For example, replacing "1_2+1_3" by "1_5" has no meaning for it does not describe the figure at all.

Students will readily draw several of the figures with four lines, such as four concurrent lines ($4_1=1_4$), a parallelogram ($4_2=4_2$), and a triangle with a fourth line on one vertex ($1_3+3_2=3_2+1_3$). These figures have code numbers of 4, 8, and 9 respectively. Once the students get this far, this immediately leads to several questions. What is the smallest code number for figures with four lines? What is the largest code number? Are there more code numbers in between? The answers to these questions give students an understanding of how complex a figure is, as well as a means of conveniently expressing this idea. In addition students learn that every figure is described by an equation, but that not all equations describe a figure.

Thus far the analysis of complex figures has used equations with numbers only. Most of the equations which the geometry student has encountered in algebra involve letters, or variables. This knowledge can be used to extend the analysis of complex figures by means of "code formulas." The following equation is an example of a code formula.

$$1_n+n_1=n_2$$

If "n" is a variable ranging over the natural numbers, then substituting values for "n" gives code equations which describe a "family" of figures in geometry. The figures for this formula are a transversal intersecting any number of parallel lines. The reader may be interested in drawing figures for the following formulas.

$$1_n+n_2=1_n+n_2$$

$$2_{(n+1)}+n_2=(2n+1)_2$$

The simplest formulas describe concurrent lines ($n_1=1_n$) and parallel lines ($n_0=0_n$). A rather complex formula is for a set of projective figures, that is, figures in which each line intersects every other line and no three lines are concurrent.

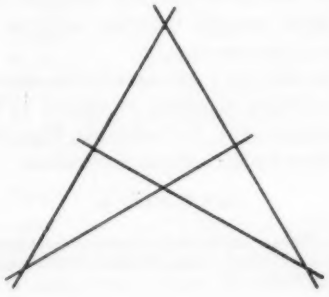
$$(n+1)_n=\left[\frac{n(n+1)}{2}\right]_2$$

If $n=3$ is substituted in the formula then this gives the equation $4_3=6_2$, which means that there are 4 lines, each of which is on 3 points of intersection, and there are 6 points of intersection, each of which is on two lines. This describes the Figure 2.

In the process of writing equations for figures, some of the students will discover that two figures have the same code equation and yet the figures are not exactly alike. For example, they may discover that the Figures 3 and 4 have the same code equations.

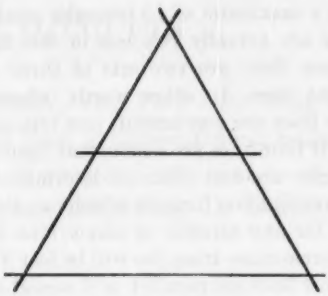
This demands an explanation, for the figures are different in one main respect. The difference is that in the "hourglass" figure the triangles do not overlap, while in Figure 4 on the right the triangles do

Figure 2



$2_2+2_3=5_2$

Figure 3



$2_2+2_3=5_2$

Figure 4

overlap. However, the incidence relations for both figures are the same. Of course figures may differ in other respects, such as one figure being rotated in a plane as compared to another figure. All these differences need to be explained to students.

The use of code equations gives students certain information which helps in analyzing complex figures. In addition, students need to have other information, such as the number of angles, straight line segments, and triangles. There is no magic formula which will provide all this information. The student's first reliance is simply locating and counting them. There is, however, a method which will help students with these problems. It is the use of permutation and combination formulas.

The number of angles at *one* vertex is the number of combinations of distinct pairs of rays which are on that vertex. For example, two intersecting lines, or four rays, form $4(4-1)/2$ or 6 angles. Likewise the number of straight line segments on *one* line can be determined by the combinations of the points of intersection. In Figure 5 the number of straight line segments on the bottom line can be found by the combination of distinct pairs of the four points of intersection. Thus $4(4-1)/2$ gives 6, which is the number of straight line segments.

There is no formula which will give the exact number of triangles in every figure. However, it is possible to find out the maximum number of triangles for a certain number of straight lines. Figure 6 has five lines.

The combination formula for five things taken three at a time,

$$\frac{5(5-1)(5-2)}{3\cdot 2\cdot 1},$$

Figure 5

$1_4+4_2=1_4+4_2$

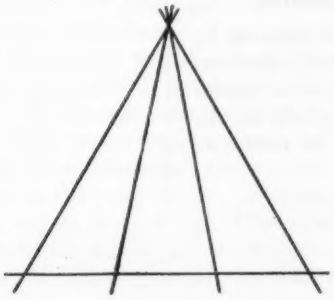
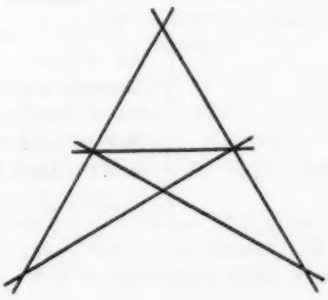


Figure 6

$4_3+1_2=2_3+4_2$

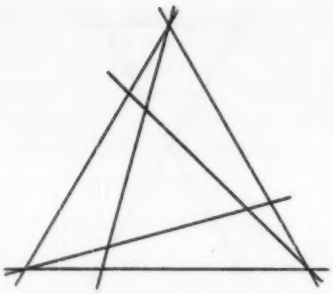


gives a maximum of 10 triangles possible. There are actually two less in this figure because there are two sets of three concurrent lines. In other words, whenever three lines are concurrent, one triangle is lost. If four lines are concurrent then four triangles are lost. This is determined by the combination formula which can also be used for any number of concurrent lines. Of course more triangles will be lost if two or more lines are parallel, or if straight line segments are not parallel but are nonintersecting in the figure. The figures which do contain the maximum number of triangles possible are the projective figures whose formula was given previously. These are the figures in which every line intersects every other line and only two lines are on any one point of intersection.

The reader may be interested in trying to *count* the number of triangles in Figure 7.

Now use the combination formula to find the maximum number of triangles with six lines. Next subtract the triangles which are lost because of the concurrent lines. This procedure gives the correct number of triangles rather easily.

These methods of analyzing complex figures in geometry have a number of advantages. (1) Students gain an understanding of the structure of figures and how a given figure is related to other figures. (2) If students are required to write the code equation for each figure used, then this may, at least with better students, help overcome the difficulty of dealing with complex figures. (3) It has been the writer's experience that most students will find this work enjoyable. Students will at least have less fear and more confidence in handling complex figures. (4) Students also learn that the language of algebra is an indispensable means of expressing geometric ideas conveniently and compactly. Needless to say, it does no harm to emphasize the harmonious relationship of algebra and geometry.



$3_3+3_4=3_3+6_2$

Figure 7

Annual Business Meeting

Notice is hereby given, as required by the Bylaws, that the Annual Business Meeting of the National Council of Teachers of Mathematics will be held at the Baker Hotel, Dallas, Texas, at 4:00 P.M., Friday, April 3, 1959.

Questions with unusual answers

BENJAMIN BOLD, *Seward Park High School, New York City, New York.*

Students like questions requiring them to demonstrate a knowledge of mathematical ideas. Here is one way to show students that conditions may not be satisfiable.

QUESTIONS ASKED by a teacher of mathematics usually have a definite answer. It is often fruitful, however, to ask questions that seemingly have no answer or infinitely many answers. I shall illustrate that point by using examples from different levels of mathematics.

Let's start with elementary algebra. We pose the problem, "Find two consecutive even integers whose sum is 18." The equation is promptly set up and the desired answer obtained. Suppose we change the problem to, "Find two consecutive odd integers whose sum is 18." This time the equation does not produce the desired answer, and further investigation proves both fruitful and stimulating. If the intellectual level of the class permits, the conditions under which the two consecutive integers will be even or odd can be determined. If even, they would be of the form $2n$ and $2n+2$. The sum would be $4n+2$, which is divisible by 2 but not by 4. If odd, they would be of the form $2n+1$ and $2n+3$. The sum would be $4n+4$, which is divisible by 4. The pupil can then acquire a better appreciation of the method used by the teacher to have the problems always give suitable answers.

An illustration from geometry is, "Find an angle whose supplement is 40° more than its complement." One possible equation is $180-n=40+90-n$, whose solution set is the null set. What does that signify? It should not be difficult to see that the above question contradicts the fact that the supplement of an angle is always 90° more than its complement.

Using two unknowns can lead to other interesting results:

If x = number of degrees in the complement and y = number of degrees in the supplement,

(1) $y=x+40$
(2) $y=x+90$

} A pair of inconsistent equations representing parallel lines.

If the question were, "Find an angle whose supplement is 90° more than its complement," an identity ($n+90=n+90$) results, and the solution set contains all permissible values of x. Using two unknowns, we obtain,

(1) $y=x+90$
(2) $y=x+90$

} A pair of dependent equations representing one line.

If we change the question to, "Find an angle whose supplement is 40° more than twice its complement," we obtain,

(1) $y=2x+40$
(2) $y=\ x+90$

} Consistent equations, different slopes, lines intersect.

If we change 40° to 100°, $x=-10°$, $y=80°$, angle = 100°, what is the significance of the fact that the number of degrees in the complement of the angle is a negative number?

A second illustration from plane geometry states, "Given line segment $AB=12''$. A point P is equidistant from A and B,

and a given distance from A. How far is P from AB?"

If we make the given distance 5″, we have

$$x^2+6^2=5^2$$

$$x^2=-11$$

$$x=\sqrt{-11}$$

What does $\sqrt{-11}$ signify? Circle and line do not meet. Algebraically there is an answer, geometrically not.

Answering that question may be a good introduction to the meaning of an imaginary number. The above illustrations suggest that when questions that seemingly have no answer are posed, investigation of the situation may provide a stimulating intellectual experience and lead directly to a more advanced concept.

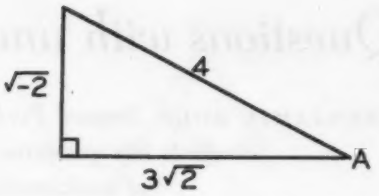
To illustrate the last point, I shall use an example from solid geometry. The problem is to find the volume of a regular square pyramid whose base edge is 6 and whose lateral edge is 4. If h is the altitude of the pyramid,

$$h^2+18=16$$

$$h=\sqrt{-2}$$

$$v=1/3\cdot 36\cdot\sqrt{-2}=12\sqrt{-2}.$$

What is the significance of the result? Under what conditions will the pyramid be real?



If b = base edge and a = lateral edge,

$$\frac{b}{2}\sqrt{2}<a$$

$$b^2<2a^2.$$

If we investigate the triangle shown above, we find

$$\text{Sin } A=\frac{\sqrt{-2}}{4}, \qquad \text{Cos } A=\frac{3\sqrt{2}}{4}.$$

Sin A is imaginary, Cos $A>1$, $\text{Sin}^2A+\text{Cos}^2 A=1$. What sort of angle is A?

Another illustration from solid geometry is, "A parallelepiped has a square base whose side is s. Each lateral face is a rhombus whose angles are 45° and 135°. Express the volume of the solid in terms of s." The altitude and volume of the parallelepiped are both equal to zero. What theorem of solid geometry is illustrated?

In planning questions, it is desirable to include occasional ones of the types indicated. For this type of question the answer is not the end of the learning situation but the beginning of a stimulating and worthwhile experience.

Say not thou, What is the cause that the former days were better than these? for thou dost not inquire wisely concerning this.—*Taken from* Eccl. 7:10.

Modern mathematics in the twelfth grade

JOE KENNEDY, *Indiana University, Bloomington, Indiana.*

The following is a description of an experimental twelfth-grade mathematics course taught by the author in 1955–1956 and 1956–1957 at Wisconsin High School, Madison, Wisconsin.

A LOT OF PEOPLE were (and still are) talking about it—but we decided to *do* something about it. After all, if modern mathematics is to be taught in the high schools to a nonnegligible share of our students in the very near future it must be taught by teachers who have not had extensive (or perhaps any) training in modern mathematics. Most of today's teachers don't know much about such mathematics and, furthermore, beginning teachers for the next few years won't know much about it either. Yet we hear a great clamoring for modern mathematics and, whether or not the clamorers are right, it does seem reasonable that some of us, here and there, ought to find out what can actually be done toward teaching modern mathematics in the high school. We thought we should, as some of my Wisconsin friends say, "Try it and see once!"

Most of us would not attempt to teach such a course without a good textbook. With the help and encouragement of Dr. John R. Mayor,* I chose a textbook (Allendoerfer and Oakley's *Principles of Mathematics*, McGraw-Hill) that seemed to contain a goodly share of modern mathematics and much of the traditional material we thought our 12th-graders needed. This turned out to be an excellent choice. Other books of this type are on the market and may be as good—this one proved to be highly satisfactory. The book was understandable, self-teaching (or nearly so), contained an ample number of exercises, and was nearly devoid of such phrases as "obviously," "clearly," "the proof is left to the student," etc., which have no place in any elementary textbook.

Our 12th-graders at Wisconsin High School had had one and two-thirds years of algebra, one-third year of trigonometry, and one year of plane geometry before entering the 12th grade. These students were well above average intelligence, as most senior mathematics classes are, and they are interested in mathematics, as such students should be. There were 15 students in the class, but 20 or even 25 could have been taught as well. The teacher had had five years' experience in high school teaching, none with modern mathematics topics.

We followed the text closely, omitting almost nothing. The first chapter is on logic. I didn't know much about logic and started the year's work with some apprehension. The students—too young to know any better—weren't afraid at all. They liked the idea of learning something altogether different from anything they had seen before and tackled the first chapter with enthusiasm—a spirit they

* Dr. John R. Mayor was then head of the mathematics department at Wisconsin High School and Acting Dean of the School of Education at the University of Wisconsin. He is now Director of Education of the American Association for the Advancement of Science, Washington, D.C.

retained throughout the course. It was pleasant teaching such a course to such a group, but it was apparent from the first that their responsibility to do homework would be accepted if and only if the teacher accepted the responsibility of preparing his work, too. Fortunately, Mr. Allendoerfer and Mr. Oakley had provided references for each chapter, which enabled the teacher to know a little more than he was trying to teach—always a safe rule to follow. By the way, the books referred to would be good ones to ask for the next time the librarian requests suggestions. Much of the material in these books could be understood by better students, but not if they never get to see it.

The second chapter was on the number system, but before any of you complain that this was supposed to be "modern mathematics" and that logic and the number system are not "modern," let me hasten to say that these topics are presented from a "modern" point of view. This manner of presentation gave us the tools and point of view necessary to study the units which followed on groups, fields, sets, and Boolean algebra. It seemed to me that while these boys and girls were learning some facts about certain mathematical ideas they were learning something else of more importance: they were beginning to understand the meaning of the postulational approach. Also, they were shaking themselves loose from the rigid "follow the rules" approach of traditional algebra and geometry. New mathematical tools gave them new freedom in their approach to problems. They were beginning to understand and appreciate the nature of mathematics. So was I.

The sixth chapter is called "Functions." My students had been exposed to functions in algebra and in trigonometry, but it was here that they learned what function meant. This unit set the pattern and laid the foundation for the more traditional work which followed. The next three chapters were: Algebraic Functions, Trigonometric Functions, and Exponential and Logarithmic Functions. You are well aware that students often do not really understand a particular topic until they are required to use it in a subsequent course or in some other application. Our students felt that these three chapters and the three which followed did teach them to use algebra, if nothing else. As a high school teacher, I found this material a little more familiar than that which preceded it; this gave me a breather before starting the three chapters on Analytic Geometry, Limits, and Calculus.

While the material in these three chapters had been included in my undergraduate work, much of it had been forgotten. Many of you who have taught only in high school might feel (as I did) that this would be a good place to stop. We couldn't, because there were 12 weeks to go in the second semester. The 11th-grade algebra work had included enough analytic geometry so that the first of these three chapters was not entirely new, although we found that some topics had not been thoroughly learned in the 11th grade. The chapter on Limits ranked along with Logic and Functions as the most important in the course. The concept of limit seemed to be the most difficult one for students to grasp, yet they finally understood it because they worked their hardest on it. If your college work on limits dealt with something called "infinitesimals," as mine did, then you too may learn much from teaching this section. The calculus chapter was taught by a student teacher, John Wood, who was then taking an advanced calculus course in the University. He found that the material we were studying was consistent with his course and yet presented in a manner which made it understandable to high school seniors.

The last chapter in the book, Statistics and Probability, was very enjoyable. Descriptive statistics, with its simple arithmetic calculations and readily grasped concepts, was quite different from limits

and calculus. After a long, tough year the class and teacher were ready for the lighter and easier problems of probability. Some of these were cookbook stuff, others were not. We had fun discussing sampling problems, such as: "How many people in Madison go to bed with their shoes on?" This isn't nearly as silly as it seems. By the time we had discussed definitions of terms, polling problems, presentation of data, and several other items we had touched on many aspects of the complex problem of sampling.

I want to be the first to say that the teacher and students were fully aware at all times that our purpose was not to learn all about groups, calculus, statistics, functions, limits, etc., once and for all and have it over with. Everyone recognized that we were taking a look at the broad field of mathematics, examining parts of it in some detail, trying to learn how previous courses and future courses were related to one another, and in many instances just aiming at getting through certain material, knowing that the second time we approached it we would find it easier. All of us felt that we were successful.

Naturally, we cannot tell yet whether or not these students will do a good job in their first two or three years of college mathematics as a result of taking this course. And we can never be sure that a different course, traditional or otherwise, would not have been of greater value. These students are sophomores in college this year. Some of them were excused from taking any college mathematics because they scored highly on placement tests. Others, who chose mathematics or applied mathematics for their college work, began with advanced standing; all but one of these made A the first semester. They have continued to maintain high standards in college mathematics courses, although they complain uniformly about poor teaching in college. Some of them have said that their nightly struggles with 12th-grade mathematics have proved invaluable, in terms of content and study habits.

A second class has now been taught this course and the third is well under way. Much was learned from the first year in regard to the ability of high school students to handle modern mathematics, regarding their tremendous enthusiasm and capacity for work, and regarding the teaching of such material by an untrained teacher.

The second year we started at a somewhat faster pace that allowed more time for the final topics. The class was not as strong as the first one, but this only meant that some sections of chapters were sketchily covered by weaker students. The book lends itself well to the omission of certain topics or even entire chapters, if necessary.

This brings me to the final point. My experience with public high schools leads me to believe that the text I mention may be used with any 12th-grade mathematics class. It may be necessary to omit some topics and emphasize others, depending on the pupils' backgrounds and needs, but this course, even in part, will be more valuable than any present course in college algebra, solid geometry, analytic geometry, or calculus. If you don't agree with me—try it once and see.

The following problems are taken from semester examinations. They are included to show more clearly the content of the course.

1. Write in "if, then" form: A sufficient condition that Lake Mendota freeze is that the temperature be below 0°F.
2. If the light is red, the car will stop.
   The car does not stop.
   Conclusion: The light is not red.
   Is this a valid conclusion?
3. Find an integer x such that $0 \leq x \leq 3$ and $x+7 \equiv 4 \pmod 3$.
4. What real values of x satisfy $6x^2+5x-4 \geq 0$?
5. Show by a series of Venn diagrams: $(A' \cap B)' = A \cup B'$.
6. Graph:

$$y = x + \frac{2}{x} + \frac{3}{5}.$$

7. State the Fundamental Theorem of Algebra.
8. The inverse of the converse is the _____.
9. Are identical sets equivalent?_____
10. In Boolean algebra (0,1) does $(1+0)\times 1=1$?_____
11. $f(x)$ and $g(x)$ are functions of x, what is $g(f)$ called?_____
12. Describe the steps used in finding $5^{3.2}$ by logs.
13. Write the equation of the line through (2, 3) which is perpendicular to the line through (7, −2) and (−1, 4).
14. $$S_n=3-\frac{2}{n}$$ is the general term of a sequence. Guess the limit of the sequence and prove that this is the limit.
15. Find first and second derivatives of $f(x)$: $$f(x)=8x^3+\frac{2}{x}-3.$$
16. Find $\int_1^5(3u^2+2u)du$.
17. Calculate: mean, median, and standard deviation for this set: 5, 3, 3, 7, 9, 4, 6, 1, 2, 5.
18. Prove that $f(x)$ is or is not continuous at $x=1$: $$f(x)=\begin{cases}\frac{2}{x}+x^2, & x\neq 0\\ 2, & x=0\end{cases}$$
19. What is the probability of getting at least two 6's in two rolls of three dice?
20. Prove by mathematical induction: $$a+ar+ar^2+\cdots+ar^n=\frac{a-ar^n}{1-r}.$$

Have you read?

HOFFMAN, H. L. "Science and Mathematics as Viewed by a California Business Executive," *California Schools*, June 1958, pp. 307–310.

Mr. Hoffman is president of Hoffman Electronics Corporation, He is not an engineer. His company employes 3,000 workers, of which over one-fourth have two or more years of technical training. He says the difference between profit and loss rests on the quality of these people. Mr. Hoffman believes that graduates are much better today than in the past, but they must be better still.

What is needed? Here are some items: more creative ability, more organizational ability and decision-making power, more willingness to deviate, more curiosity, better powers of communication.

These things cannot be accomplished except through better texts, the teaching of science earlier in school life, and more uniformity of standards. We need to lead the student to find real joy in learning and personal accomplishment.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

READ, CECIL B. "What's Wrong with Mathematics," *School Science and Mathematics*, March 1958, pp. 181–186.

This topic has been much bandied around by those who know and those who do not. For those of you without the time to analyze all that is in the literature the author has conveniently pulled together the criticisms. He suggests that the reader himself identify the person or groups that contributed each point of view, and I found this an interesting bit of homework. I didn't count the number of criticisms cited, but it appears to be quite a substantial number. Among others we find: obsolete material, excessive emphasis on manipulation, the college professor spends too large an amount of time on what the student should have learned in high school, facts of algebra are worthless if not understood, college algebra is a conglomeration, nothing new has been added to geometry since Euclid, and so on and on. I say this article is a "must" in reading for every teacher, who should be sure to read it completely.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

A unit about numbers

ROBERT M. RIPPEY, *Lyons Township High School, La Grange, Illinois.*

There are many things one can do to make a course in mathematics lively. This article offers a few suggestions.

AT EVERGREEN PARK HIGH SCHOOL, Evergreen Park, Illinois,[1] we have an advanced mathematics course for seniors, with algebra, geometry, and trigonometry as prerequisites. Although the course aims at securing advanced placement for some of its members, it has the additional aim of giving the student a survey of mathematics, interesting mathematical applications, and also some insight into the way in which mathematicians think.

I have found the following unit on numbers very stimulating to my classes, and of possible interest to other teachers because of the unusual organization of the material. In the unit, the students find several interesting surprises and have the opportunity to participate in some of the adventures of mathematical discovery that they have previously known only through reading.

We use two texts in the course, *Principles of Mathematics*[2] by Allendoerfer and Oakley, and *Introductory College Mathematics*[3] by Milne and Davis. This unit uses the text by Allendoerfer and Oakley as a key reference, but students do much of the discovery on their own, and also with the assistance of such books as *Elementary Concepts of Mathematics* by Jones.[4]

The unit begins with a discussion of the uses of numbers. The students are reminded that numbers are used for counting and for identifying objects, but that most work is concerned with binary operations, such as addition, subtraction, multiplication, and division. The operations of multiplication and addition are examined carefully with the sets of natural numbers, integers, rational numbers, real numbers, and complex numbers. In each case the question is asked, "Does this number set and operation correspond to the commutative group axioms?"

These axioms are as follows: Given a set, $S=\{x, y, \cdots\}$, and a binary operation, $*$, which combines each pair of elements into a single element, $(x*y)$, with the following properties:

1. For each x and y in S, $x*y$ is in S. (Closure.)
2. For each x and y in S, $x*y=y*x$. (Commutative.)
3. For each x, y, and z in S, $(x*y)*z=x*(y*z)$. (Associative.)
4. There is an element, e, in S such that $e*x=x$ for each x in S. (Identity).
5. For each x in S, there is an element, y, in S such that $x*y=e$. (Inverse.)

Once the student has grasped the meaning of these axioms, through the use of examples taken from arithmetic, he is asked to check them with tables and with other sets. Tables 1 and 2 are examples of the types that the student examines.

In studying these two tables, it should be noted that the identity element in Table 1 is the letter E and that the identity element in Table 2 is the letter A. The set represented by Table 1 satisfies

[1] The author taught at Evergreen Park High School before coming to Lyons Township High School.

[2] C. B. Allendoerfer and C. O. Oakley, *Principles of Mathematics* (New York: McGraw-Hill Book Co., 1955).

[3] W. E. Milne and D. R. Davis, *Introductory College Mathematics* (Boston: Ginn & Company, 1941).

[4] Burton W. Jones, *Elementary Concepts of Mathematics* (New York: The Macmillan Co., 1947).

E	A	B	C	D	E
D	E	A	B	C	D
C	D	E	A	B	C
B	C	D	E	A	B
A	B	C	D	E	A
Δ	A	B	C	D	E

TABLE 1 (deltition)

E	E	E	E	E	E
D	D	C	B	A	E
C	C	A	D	B	E
B	B	D	A	C	E
A	A	B	C	D	E
θ	A	B	C	D	E

TABLE 2 (thetaplication)

all of the group axioms. Note that the identity element for deltition is not the same as the identity element for thetaplication, and that E has no inverse in the thetaplication table. Another axiom—the distributive axiom—can also be shown to hold for these tables. It can be illustrated as follows:

$$B\theta(C\Delta D) = (B\theta C)\Delta(B\theta D)$$
$$B\theta B = A\Delta C$$
$$D = D.$$

But:

$$B\Delta(C\theta D) \neq (B\Delta C)\theta(B\Delta D)$$
$$B\Delta B \neq E\theta A$$
$$D \neq E.$$

Furthermore, it should be pointed out that when the identity element (E) from the deltition table is deleted from the thetaplication table, we are left with a set of four elements and an operation which also satisfies all of the group axioms.

The student then experiments with these two tables and discovers that he can operate with them in his usual manner—with real numbers and addition and multiplication. For example, he can solve for X the following equation:

$$(B\theta X)\Delta C = D.$$

Solution: Delt the deltative inverse of C on both sides of the equation.

$$[(B\theta X)\Delta C]\Delta B = D\Delta B$$
$$(B\theta X)\Delta(C\Delta B) = D\Delta B \qquad \text{(Axiom 3)}$$
$$(B\theta X)\Delta E = A \qquad \text{(Axiom 5)}$$
$$E\Delta(B\theta X) = A \qquad \text{(Axiom 2)}$$
$$B\theta X = A \qquad \text{(Axiom 4)}$$

Thetaplicate both sides of the equation by the thetaplicative inverse of B.

$$C\theta(B\theta X) = C\theta A$$
$$(C\theta B)\theta X = C \qquad \text{(Axiom 3)}$$
$$A\theta X = C \qquad \text{(Axiom 1)}$$
$$X = C \qquad \text{(Axiom 4)}$$

Check.

$$(B\theta C)\Delta C = A\Delta C = D.$$

The student also tries solving simultaneous equations using the same tables.

Here, then, the student finds a system which, as nearly as he can tell, is much like the familiar set of postulates for real numbers. He is next asked to seek out the similarities between this set of postulates and the set of real numbers to see if there is not a list of properties that will characterize this set and its two basic operations.

First, it is obvious that this set forms a commutative group with respect to the first operation. Secondly, when the identity element with respect to the first operation is excluded from the set, it also forms a commutative group with respect to the second operation. Thirdly, the distributive law holds. The set is closed with respect to both operations.

The class also discusses some of the interesting theorems concerning primes, such as the infinite number of primes, the possibility of an infinite number of twin primes, how to determine whether a number is prime (including some short cuts), the distribution of primes, the number of primes between 1 and an arbitrary number, and, finally, the possibility of a formula for determining whether a number

is prime or not. Several examples are given, after which the students are instructed to invent their own formulae for primes and then show the point where they break down. A few students, of course, invent formulae that break down on the first step, but others have turned in unique formulae that have been too tedious for anyone to break down. These bring up the question of proof: How do you know whether a formula will break down or not?

At this point I perform a few feats of mental wizardry. This part of the instruction involves a bit of acting on the teacher's part. The work has to be timed in such a manner that the following exercise is completed five minutes before the period ends.

First, I ask for some student, respected by his classmates for his calculating ability, to assist me. I call for two numbers at random from the class, my only request being that these numbers be less than 100. Next, I tell the student at the board to write down the two numbers and to add them. Finally, the student is told to continue adding each result to the number just preceding it until he has ten numbers. This, of course, results in a Fibonacci series of ten terms. A typical series is:

35
7
42
49
91
140
231
371
602
973

I tell the student to tell me when he is finished. Before he gets a start at adding these ten terms, I turn around and write the sum (2541) on the board. I then tell the students that I want them to practice until they can do the same thing I did, and that this will be their next day's assignment. Before the bell rings, I admit that there is a trick to this feat of mine, and that if anyone discovers it, he should try to prove why it works. I also give them one hint that might be helpful—I suggest that they find the prime factors of the answers to several of these series.

By the next day, most students have discovered the trick. They find that all answers have a prime factor of 11 and that 11 multiplied by the fourth term from the bottom gives the result. A few already will have the proof. And when I point out that the series can be written as a, b, $a+b$, $a+2b$, etc., most of the rest can finish the proof in class.

So far my classes have not disappointed me by failing to ask the next key question, "Do you know any more tricks like that?" My answer is to add all the numbers from 1 to 1000, or to make some other calculation based on series formulae. I then derive these formulae inductively on the basis of a few examples, and the stage is set for an introduction to induction. We recall our experience with prime number formulae and find that, convincing as the formula N^2+N+11 appeared, it did have a point where it no longer held. Here we have developed a desire to use the formula $n(n+1)/2$, yet we have also raised questions as to its applicability to any number whatsoever. Induction in this form appears to the student as a logical necessity, not merely as a requirement for graduation.

While proving the formulae for the more important series, we introduce arithmetic and geometric progressions, convergent infinite geometric series, and the problem of changing a repeating decimal into its equivalent fraction. We then finish the unit with a search for patterns in series. We work out the series for:

$$1^1+2^1+3^1+\cdots+N^1 = \frac{N^2}{2}+\frac{N}{2}=\frac{N(N+1)}{2}$$

$$1^2+2^2+3^2+\cdots+N^2 = \frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}=\frac{N(N+1)(2N+1)}{6}.$$

On the basis of these formulae, which the student is asked to check by induction,

the students are asked to propose formulae for the sum of the cubes of a series of consecutive integers. A commonly proposed one was:

$$\frac{N(N+1)(2N+1)(3N+1)}{24},$$

which was shown to fail.

However, when the students were given the true formula for the series,

$$1^3+2^3+3^3+\cdots+N^3 = \frac{N^4}{4}+\frac{N^3}{2}+\frac{N^2}{4}=\frac{N^2(N+1)^2}{4},$$

they felt that there must be a pattern to these formulae and became interested in discovering it. Some worked on the problem; others anticipated its difficulty and left it for their colleagues. It is probable that the formulae for the fourth and fifth powers will be discovered by at least one enterprising student in a class. However, the seventh powers and beyond are more elusive. The complete solution for all powers of the integers was first worked out by Jaques Bernoulli in his *Ars Conjectandi* in 1713.

It may seem unfair to stop a unit at this point. Yet the student has had a glimpse of how one generality may be contained within another. He has had a chance to make new discoveries, contained in an acceptable solution to a less challenging problem. From these experiences, the student may better understand the mathematician's persistent desire for greater simplicity and generality.

The methods in use in schools ought . . . to differ from the Newton, Darwin, Faraday methods of private study only as the little sapling, which still needs support and shelter, differs from the trees.—*Taken from* The Preparation of the Child for Science *by M. E. Boole.*

Mathematics is no more the art of reckoning and computation than architecture is the art of making bricks or hewing wood, no more than painting is the art of mixing colors on a palette, no more than the science of geology is the art of breaking rocks, or the science of anatomy the art of butchering.—*C. J. Keyser.*

General mathematics for terminal students in California junior colleges

JACK L. ROWE, *Bakersfield College, Bakersfield, California.*

A report on the efforts of one junior college to improve the mathematics curriculum. How to stimulate an interest in mathematics *at this level needs further study.*

THE JUNIOR COLLEGE is fast becoming a recognized institution which provides opportunities for students to continue their formal training beyond the high school. The mathematics curriculum in many of these schools might be improved. In particular, the mathematics program for terminal, nontechnical junior college students is deficient. The purpose of this study is to develop and evaluate a course in nontransferable general mathematics for these particular junior college students. A questionnaire organized to select objectives, content, and organization of the course was developed and validated by a jury of 16 experts in the fields of education and junior college and university mathematics. Careful study of the recommendations and findings of previous investigations, reported in the literature during the past few years, influenced the construction of the questionnaire. The instrument was sent to three groups of people directly concerned with general mathematics—all the mathematics instructors in the California junior colleges (175 in number), 198 terminal junior college students in two California junior colleges, and 150 recent graduates of a California junior college.

Analysis of responses to the questionnaire was used to select the content and objectives of the course, which were developed into a syllabus used as the basic text for the course. Briefly, the main topics included in the general mathematics course were: review of fundamental processes in arithmetic, including fractions and decimals; percentage and its application to business and living; business and consumer concepts of discount, installment buying, banking, investments, income taxes, social security, and insurance; budgeting; graphs; elementary algebra; formulas and their applications; ratio and proportion; units of measurement and their conversions; estimation and approximation; critical and constructive interpretation of problems; and an awareness of the role of mathematics in the world today.

This course was taught to two experimental classes of terminal students in a California junior college for one semester. The classes were organized to meet twice a week for a one-hour lecture and once each week for a two-hour laboratory period. An equivalent group of students enrolled in a business mathematics course and a similar group of students not taking mathematics were selected for control groups. These groups were made equivalent by matching insofar as possible the quantitative scores students made on the Cooperative School and College Ability Test.

Conclusions resulting from measures of evaluation were:

1. Responses on the questionnaire indicated a need for such a general mathematics course.
2. Statistical analysis of scores on a specially constructed mathematics test given to all groups indicated significant gains in achievement for the experimental group in comparison with the two control groups.
3. Continuous student and instructor evaluation of the course indicated that it had considerable merit in providing terminal students with practical and useful experience in mathematics.
4. Terminal, nontechnical students in junior colleges are able to learn an impressive amount of mathematics in spite of allegations to the contrary by many.

Considering the favorable results for this study, specific recommendations are:

1. This course in general mathematics would be a valuable addition to the curriculum of the junior college.
2. There should be a comprehensive screening examination in mathematics given to all matriculating junior college students. Those who fail to reach a minimum proficiency on this test should be required to enroll in a course in general mathematics similar to that proposed in this study.
3. Credit for the general mathematics course should not be transferable to a four-year college or university awarding the A.B. degree.
4. Additional study should be conducted in a variety of situations to confirm the apparent merits of the course in general mathematics suggested in this study.

Math Rules the World

Black soldiers marching on a field of white
They battle on, to down the foe
Or calculate how many crops will grow
And when the sun will rise
To rule the world.

Soldiers guided by a general's hand—
Who shall he be?
A strong, a valiant man, a genius, or a fool?
Choose well
For he will rule the world.

Soldiers blessed with talents,
They tell of time, survey the land
When guided by the general's hand.
Without it they are lost
And yet they rule the world.

An army of numbers,
Symbols, on a paper battleground,
Yet it is far beyond that their victories are found.
Math's master is a mighty man,
He rules the world.

—*Barbara Bond (11th Grade Student), University of Minnesota High School*

Symbolized theorems

MERIAM A. CLOUGH, *Cupertino High School, Sunnyvale, California.*

An interesting device for symbolizing theorems is presented here. It should not, however, lead to overemphasis on writing detailed proofs throughout the year of geometry in the secondary schools.

DURING THE PAST FOUR YEARS I have found one simple device more and more helpful in the teaching of geometry. This device is a method of abbreviation which students may use when stating reasons in a geometric proof. At first I used only a few abbreviations in my classes, but now I find many postulates, theorems, and corollaries can be adapted to the method. And the students will often improve my abbreviations or invent new ones I hadn't suggested.

Basically the method is one of stating a theorem or postulate in the "If . . . , then . . . " form by use of symbols. For instance, the theorem "If two sides of a triangle are equal, then the angles opposite these sides are equal" may be reduced to

"If , then ,"

(with similar marks indicating equal parts). The converse is very easy to write

"If , then ."

In this manner, the theorems on parallel lines can be worked out very simply. "If two lines are parallel to each other, the alternate interior angles are equal" can be reduced to

"If , then ."

The rest of the parallel theorems and their converses follow.

In a similar fashion the theorem "The opposite sides of a parallelogram are equal" can be symbolized as

"If , then ."

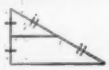
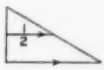
The more complicated theorems can also be abbreviated. For instance, "If a line passes through the center of a circle and is perpendicular to a chord of the circle, then it bisects the chord and its arc," can be written as

"If , then ."

Likewise, "If a line bisects the vertex angle of an isosceles triangle, it is perpendicular to and bisects the base" becomes

"If , then ."

"The line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to one half of it" can be symbolized (with arrows indicating parallel lines) as

"If  , then  ."

Use of these abbreviated forms does not excuse the student from reading, understanding, and proving a theorem. Only after such a proof has been made should the teacher and students, following carefully the meaning of the theorem, work out abbreviations to be used.

One of the great advantages of using symbols as described above is the clarity and brevity of the abbreviations. Even the poorest student can see what a theorem means and begin to use it at the right time in his proofs. Sometimes during the early part of a chapter or unit it helps to have all the theorems of the chapter to date listed on the blackboard in symbolized form for the students to use as a reference while doing their homework. Then a student in difficulty can be asked to point out which theorem might help solve his problem. Students will often make lists for themselves to use in the same fashion. Since the basic ideas of a theorem are all pictured, the pupil can often find what he needs at a glance; he can also see that the theorem really fits the particular situation in his problem.

Another advantage of symbolized theorems is their value in reviewing a chapter or unit. The class can help group theorems of any chapter in two or three large categories. For instance, the theorems concerning parallel lines can be grouped in three categories. One group could start with

"If , then ."

The second group would then start with

"If , then ."

And the third group would be headed

"If , then $3\measuredangle = 180°$,"

followed by various corollaries. Sometimes there are miscellaneous theorems which can be fitted in or be considered as a group by themselves.

If the teacher gives closed-book tests, of course lists of theorems in symbolized form cannot be used during the tests. But after using theorems in this form throughout a unit or chapter, frequently having them on the board, and grouping them for review, a student should be able without too much difficulty to recall theorems and use them intelligently during a test.

Function

A box with a funnel sticking out,
At the opposite end an elbow spout,
Inside the box some works, no doubt.

The set of x's is used to fill—
Food for machine, grist for the mill.
One y for each x the spout will spill.

Set X—the domain. That's not too strange.
The rule (the box) effects the change:
x into y. Set Y is the range.

The concept's clear—no mental blocks.
Now this may appear a paradox,
But that is the function of the box.

—*Katharine O'Brien*

*Freshman mathematics in the junior college**

THEODORA R. QUICK, *Grand Rapids Junior College, Grand Rapids, Michigan.*

America's educational system provides a variety of educational programs of which the city junior college is a significant unit. This article describes the mathematics program and presents the educational philosophy of one junior college.

IN OUR OWN INSTITUTION, as in many others, the advent of the veteran on the campus resulted in a minor revolution in our educational and curricular planning. Many of us, I am sure, remember this postwar period with amusement. Some of us dreaded the predicted invasion of these young men, whom we pictured to ourselves as hard-bitten, hard-living young adults, little amenable to the disciplines of learning.

We were happily surprised. Here, indeed, was maturity; but it was maturity with a driving purpose. Those veterans who chose college came to us with the intention of making up for lost time. Their years in the ranks made them determined students, and most of them set a rugged pace for their classmates who came directly from high school. But their determination too often was mixed with dismay when they learned that in certain subjects (of which mathematics was one), they were quite unprepared for college work.

Obviously, it was necessary to establish special courses designed to supply the needed preparation, courses to provide the needed background in the least possible time. These courses remain in our curriculum, although the tide of veteran freshmen has almost ceased. One may ask whether the existence of high school mathematics courses in college does not encourage the neglect of these subjects in high school.

Let us review the problem, as we find it in Grand Rapids Junior College. Students entering our institution as freshmen in September, 1956, had a choice of four mathematics courses, depending on their preparation. Starting at the very beginning of high school mathematics for 465 freshmen mathematics students, these courses included:

1. *High School Algebra 1 and 2*	41 students
High School Geometry 1 and 2	52 students
Total:	93 students

Each of these classes, to cover an entire year's work at the high school level, meets three times weekly for a single semester, with 48 to 50-minute periods. In this short time we try to cover work which is taught in 175 to 180 high school class periods. (Credit, of course, is seldom transferable.)

* The original version of this paper was written to place some of the problems encountered in our Grand Rapids Junior College mathematics department before a seminar of high school teachers.

Today, however, most of the students in such classes are only two or three months older than when they graduated from high school. Many lack the incentive and application we found in our veterans. Indeed, many were high school "drifters"; many did not plan to enter college; and many had no concept of college preparation. Some admit they avoided subjects requiring concentrated study. While we must accommodate these people, there are several unfortunate results:

a. Greatly added expense to the individual and his family. Inadequate preparation usually means an additional year in college and loss of a year of adult productivity.
b. Diversion of time and effort of instructors; overcrowding of true college mathematics classes; over-busy instructors, with resulting sacrifice of personal attention for those who deserve it.
c. Added cost for the taxpayer.

For these reasons, these high school courses will probably be taught in the future in the evening only by competent high school instructors.

2. *Solid Geometry and Trigonometry:* 92 students

Classes meet twice weekly. For the full semester, due to interruptions, we have been able to provide about 30 periods to cover material which, in high school, would have required 85 to 90 class periods.

3. *High School Algebra 3 and Trigonometry:* 141 students

About 180 class periods would have been devoted to this work in high school. We must cover the ground in about 62 periods, one third as many. Including those students who drop the course because of impending failure, we suffer an average of 50 per cent failures among students who elect this course. Those who earn a passing grade, however, usually receive transfer credit.

4. *College Algebra and Plane Analytics:* 139 students

This is the one course open to freshmen which can be said, without apology, to provide college-level mathematics. Students electing "Math 13" have usually taken all the mathematics available in their high schools. Last semester, of a total enrollment of 139 students, 50 did excellent work. On the other hand 37, more than one fourth, did unsatisfactory work. On the whole, those who earned an A or B in this course entered college with A or B averages in high school mathematics. In a few cases, doubtless because of increased maturity (as with veterans), or because of some new incentive, students who brought a C minus average worked hard, overcame their handicaps, and earned excellent grades. Unfortunately, the reverse was sometimes true. Among the 37 who did unsatisfactory work, earning D or E grades for this course, 11 had A or B high school averages, and 9 had C plus averages. Simply stated, 20 of 37 failures, if judged on the basis of high school grades, should have been able to succeed in "Math 13" with only a fair amount of application.

Let me restate the problem. The courses described represent the total of mathematics available to entering freshmen. Of 465 freshmen who elected these courses, only 139, or 30 per cent, were prepared to enter a mathematics course of true college level. The rest—70 per cent—simply are not ready for college mathematics. These must take high school work at a considerable sacrifice to themselves, their families, and the taxpayer.

What can be done in a positive way to remedy this defect in preparation? Our own mathematics faculty has discussed the problem at length. We have gone to the students themselves, with encouraging results. Each student in our mathematics classes has given us, in written answers on a questionnaire form, his own opinion and suggestions, based on his personal high school and junior college experience. A re-

view of these papers has led us to the following suggestions and recommendations for students hoping to succeed in college mathematics:

1. Students should take all the mathematics their high school provides.
2. Daily assignments should be given and deadlines met. At least one difficult problem should be handed in daily. This problem should be worked out in a neat, orderly, readable, and logical manner. Daily work should be graded, recorded, and returned to the student.
3. Short daily tests. If a student expects to be tested daily on some single problem during the last few minutes of the class period, it keeps him on his toes.
4. A full hour's test at least every two weeks.
5. A systematic, realistic, and not too liberal method of grading, so a student can keep his own record.
6. A mathematics notebook can be a great help. Most freshmen are amazed at the idea of a mathematics notebook, and many of them are equally surprised at its usefulness.
7. A student's ability to express himself orally and in writing certainly is worthy of cultivation. Eventually, this should lead to expression of one's concepts in mathematical terms. After all, mathematics *is* a form of expression—a language, if you please.
8. This final point is constantly stressed by students: Any concept in hand must be presented in such a way that the mathematical reasons and principles involved are made crystal-clear. Every teacher is aware of the great simplicity of basic mathematical laws, such as the axioms, and of their wide application. It is important, I believe, to emphasize this basic simplicity to the student—to make him fully aware of the fundamental nature of these laws. If this mental and logical approach can be encouraged, it will help do away with a too-frequent dependence on meaningless mechanical and manipulative skills. The latter may often confuse the student at a time when logical thinking is essential.

We all are aware of the vigorous call for capable students to major in courses such as science and mathematics. No one can deny that too many talents in these related fields are going to waste. In this world and in this century, our Western culture and civilization, indeed our very existence, may depend on the conservation and nurture of these talents. There are many, I know, who deplore this emphasis on "technological" subjects, and who see in the humanities a field of knowledge equally important to the future. I believe, for myself, that both points of view can be satisfied by a relatively simple measure.

That measure, in its essence, is an attempt to discover, as early as possible in the educational process, those youngsters who are capable of higher education (in whatever field) and an all-out effort to see that they are properly prepared to make full and effective use of their natural bents and capacities. If we are to call ourselves educators, we can do no less.

How soon should we start to test for aptitudes? Not later, I believe, than entrance to high school. On the European continent, students are sorted out and categorized even earlier. Most of us do not believe in setting a child in an educational strait jacket at the age of eleven. But we should know who has the capacity for higher education. If posssible, we should know which children have a gift for mathematics, science, languages, music. And with all our hearts and all our persuasion, we should try to see that these gifts are developed and given opportunity.

Under this point of view, any student of average ability or better should be guided toward an academic rather than a vocational high school curriculum, one leading definitely toward college entrance. Only in this way can we avoid the present appalling waste of native ability—and, instead, turn that ability into creativeness.

There is no lack today of applications for college acceptance. With an oversupply of applicants, good colleges are stiffening their entrance requirements, even as they are increasing their rates. More and more, they find it impossible to spend money, space, time, and teaching effort on students who need basic work that once was essential for matriculation.

How can the individual high school teacher deal with this problem, until his superiors are ready to plan, finance, and integrate programs of guidance to encourage and lead the able student?

First of all, he can keep his eyes open

for ability and give it all his encouragement and counsel.

He must work his students, and make them like work. They should be given classwork to the limit of their capacities, and trained in study methods to match.

A way must be found to explain to the parents what is being done and why. Home co-operation is essential. Parents must be informed as to what curriculum will get a child off to a flying start in college. They should understand that their own interests may be affected by the subjects elected both in high school and in college.

There is a place in this program, once it meets some acceptance, for the community, as well. I am thinking of the encouragement and guidance to be obtained from members of some professional groups and businessmen. Many of these men recognize the problem here set forth; some know from harsh experience the penalties imposed by inadequate preparation.

Our school is fed, principally, by five city public high schools, by two very large local parochial high schools, and by eight or ten suburban high schools in our metropolitan area of about 300,000 people. Since we are a part of the city's public school system, every high school student who graduates is welcome, just as long as he does the work we require, and pays the very modest fees.

A fair proportion of those who enroll may elect two-year terminal courses. Many, however, come to us with this challenge:

"Here I stand, in your classroom, a new member of your freshman class. No matter what preparation I have had, no matter what courses I have taken or how well I have mastered them, you have two years in which to make me an acceptable member of the junior class of *X* college or university."

It is no secret in our community that students work hard at Grand Rapids Junior College and that many of them make admirable records when they reach their four-year school. If we appear rigorous in our standards of accomplishment, it is because of the challenge every one of these youngsters presents. The college to which he aspires may have entrance requirements considerably more strict than those we can enforce. If he is to enter that school without loss of credit, it is because those who have gone before him have done well, as scholars and as good citizens.

The report [Educational Policy Committee, NEA] warned against a flood of misinformation which is sweeping across the country comparing the Soviet and American systems. It noted that in the highly touted ten-year elementary and secondary Soviet system only half of the students enter the last three years and only one-third graduate.—*Educational Policy Committee, NEA.*

"Mathletes"

GEORGE LENCHNER, *Valley Stream North High School, Franklin Square, New York.*

Here is a report of the activities of the Nassau County Interscholastic Mathematics League.

ALL PENCILS DOWN. Turn your copy of problem 1 face-up. "The refreshment expense at a party was $5.29. Each boy was required to pay an equal amount to defray these expenses. It happened that each boy paid for his share with 5 coins. How many dimes were collected?" Time: 5 minutes.

The above problem initiated the 1957–1958 third annual competitions of the Nassau County Interscholastic Mathematics League. Two years ago, the NCIML came into being to provide stimulating and enriching mathematical experiences for high school students. At the initial organizational meeting, faculty representatives from 18 high schools of Nassau County convened. Some were in agreement that the excellent contests of Pi Mu Epsilon and the American Mathematical Association were inadequate for maintaining a high level of interest throughout the school year and felt that the League might fill the gap. Some raised questions concerning the educational value of such competition, teacher-load, conflict with mathematics clubs, schools with small enrollments, new high schools with only tenth graders, and the like. Nevertheless, the 18 schools agreed to try the League and evaluate it after the first year's experience.

OPERATION

The League is patterned somewhat after a sister League in New York City. Some of the features of the NCIML follow.

1. Member schools train a squad of "mathletes" who compete against other squads in 5 contests during the school year.

2. Each contest consists of 5 problems which are individually posed with a time limit specified for each.

3. A school may enter a maximum of 5 mathletes for each problem. Substitutions may be made without restriction prior to the posing of the problem.

4. For each correct answer, one point is credited to the individual mathlete and also to his team score. Scores are cumulative for the 5 contests.

5. Member schools meet in a group of 3 or 4 schools for each contest. The group is determined by geographical proximity with the role of host rotated. Problems are mailed to the host school about one week prior to the contest.

6. After the conclusion of the annual competitions, awards are presented as follows: trophies to the 3 schools with the highest cumulative scores; medals to the mathletes of the 3 schools; trophies and special awards to the 3 highest-scoring mathletes of the League; a book to the highest scorer of each individual team; silver "League Pins" to mathletes scoring 5 or more points in competition; certificates to all participants.

SPONSORS

The impressive awards are made possible by the financial assistance of sponsors. Leading industrial and engineering firms of Long Island are invited to sponsor the awards. Last year, Arma, Fairchild, Grumman, Republic, Servomechanisms, and Sperry were sponsors of the League. It is hoped that, by increasing the number of sponsors this year, a modest scholarship program will be initiated.

Authors

Dr. J. J. Kinsella, Chairman of the Mathematics Education Department of New York University, authored and compiled the problems for 1955–1956. Mr. H. Sitomer, Chairman of the Mathematics Department of New Utrecht High School, Brooklyn, New York, wrote the problems for 1956–1957. The author mails his sets of problems to an individual who then edits, mimeographs, and mails the problems to the teachers-in-charge at the host schools.

Evaluation

At the end of the first year's competitions, the faculty sponsors met as originally planned. They were in agreement that the League had fulfilled its stated objective and that the individual doubts had been dispelled. The comments were revealing and unexpected in many cases. In one school, the mathletes met after school hours to conduct their own practice when the coach was not available. A faculty mathlete coach addressed his school board on the League and its functioning. Some school publications featured the activities of their mathletes in lead columns. A principal included the standing of the Math Team among those of the athletic teams in his report to student assemblies. Some nonfunctioning math clubs were revived and revitalized. The local press carried articles about the League. Three new schools with tenth graders found that their mathletes were imbued with enthusiasm for mathematics although they had not fared well in the final standings. Some groups had socials with refreshments after each competition.

The growth of the League has been unusual. In the first year, 187 mathletes from 18 member schools had participated in one or more problems. Last year, 237 mathletes, representing 23 schools, competed. For the current year, 33 schools have entered teams in the competitions. This rapid growth indicates a need for such a challenging and competitive activity for high school mathematicians. In essence, the League is an additional means for providing enriched mathematical experiences for superior students.*

Problems

The following are some of the 25 problems from the 1956–57 competitions:

1. Time: 6 minutes. Two chords of a circle, AB and CD, intersect at E. $AE:EB = 2:3$ and $CE:ED = 1:4$. Which two of the following must be true? *a)* $AB > CD$ *b)* $AB = CD$ *c)* $AB < CD$ *d)* $AE:ED = \sqrt{6}:6$ *e)* $EC:EB = 6:1$

2. Time: 3 minutes. Find the smallest positive integral value of X that will make the product $1260X$ a perfect cube.

3. Time: 7 minutes. Alice stands 12 feet from the base of a street lamp. Beatrice, who is as tall as Alice, stands at the end of Alice's shadow and casts a shadow that is twice as long as Alice's. How many feet long is the shadow cast by Beatrice?

4. Time: 8 minutes. Two sides of a parallelogram are 7″ and 9″. If one of the diagonals is 8″, how many inches are there in the other diagonal?

5. Time: 7 minutes. Ann gave Bess and Cora each as many cents as each had already. Bess then gave Ann and Cora each as much as they each had then. Then Cora gave Ann and Bess each as much as they each had then. They ended with 16 cents each. How many cents did Ann start with?

6. Time: 3 minutes. An integer is between 20 and 30. If it is added to its cube, the sum will be 13,848. Find the integer.

7. Time: 7 minutes. A reporter asked 205 people if they liked candy, cake, or ice cream. Each person said he liked at least one of them. If 110 said they liked at least candy, 115 said they liked at least cake, 30 said they liked only candy and cake, while 10 said they liked all three, how many said they liked only ice cream?

* Further information may be obtained from the current president, Ezra Reed, Baldwin High School, Baldwin, New York. Copies of the constitution and last year's problems may be obtained from the secretary, Elmer Heinecke, Wantagh High School, Wantagh, New York. Please remit $.10 to cover costs of mailing and mimeographing.

*The nature and content of geometry in the high schools**

JULIUS H. HLAVATY, *De Witt Clinton High School, New York City.*

The nature of the geometry course is being scrutinized carefully by the Secondary School Mathematics Group. Furthermore, the Commission on Mathematics has made its recommendation on geometry. What will the geometry program be in 1969?

AFTER GEOMETRY had completed its long odyssey—from the discussions of the Greek philosophers and the school of Alexandria, through the quadrivium of the middle ages, to the universities of the nineteenth century—it finally settled down in the tenth grade of twentieth-century high schools in America. There it became the backbone of mathematical instruction in secondary schools. That it should have assumed so important a place is not surprising. Even a poor textbook in geometry—based as it was by the accident of history on the great model of Euclid—gave the high school student his first massive contact with a basic method of mathematics. Trigonometry and solid geometry were but extensions of geometry, and algebra was generally treated as a poor relation—as a hodgepodge of unrelated facts and tricks which had little, if any, relation to the great methods of mathematics.

When the implications of mass education began to emerge in the 20's and 30's, it was to be expected that the subject hardest hit by the revolution would be geometry. Vast numbers of students showed themselves either uninterested in or unable to cope with mathematics, and especially geometry. With the eternal and glorious optimism of our profession, we thought the answer lay in improving our methods of teaching and our choice of materials. At the ninth-grade level we introduced courses in general mathematics, consumer mathematics, shop mathematics, mathematics for life. At the tenth-grade level we taught practical, laboratory, straight-thinking geometry. We sugar-coated the subject matter of geometry, we streamlined methods of proof, we removed from the course all the hard topics.

Let me make it clear that I am not sneering at these efforts. They were truly heroic attempts to wrestle with overwhelming problems—problems which are nowhere near solution even today. But the fact remains that some of the consequences were little short of tragic. For example, while the removal of the "hard" topics made the subject easier, it also made it less meaningful. It approaches educational irresponsibility to remove a topic that is "hard" merely because it is hard. We would not dream of attempting to prevent a child from learning to walk merely because the process is often difficult and even painful. What the situation required was a critical evaluation of what mathematics is and of what the schools might contribute to teaching it—whether or not it was hard. Happily, we are in the midst of just such a reconsideration.

* An address delivered at the annual convention of the Council, Cleveland, Ohio, April 13–14, 1958.

Let us come to our subject. What is geometry and what are the responsibilities of the secondary schools with respect to it? Geometry, as a branch of mathematics, is a pure deductive theory, that is, a collection of sentences, some of which are chosen as postulates and the rest of which are deduced logically from those postulates. The sentences contain words, some of them taken as primitive terms (for examples, "point," "line," "between," etc.) and then used to define the others. A pure deductive theory, then, has structure but no content. The primitive terms are variables, and the sentences, therefore, are open sentences.

If geometry has only structure without content, what distinguishes it from other deductive theories? Is geometry merely a branch of algebra because it is only a study of sets of ordered sets of numbers? Is it a branch of topology because it is concerned with notions of limits, continuity, and neighborhood? These seem to me to be questions for the graduate schools and for metamathematics.

Historically, Euclidean geometry has not been an abstract deductive theory in the sense in which we use that phrase today. To generations upon generations of students, teachers, mathematicians, and philosophers—from Euclid to Lewis Carroll and beyond—geometry has been a model of a deductive theory in the sense that the primitive terms were thought to have meanings and the sentences were thought to be true statements. I am persuaded that to Euclid "point" and "line" were real—at least, imaginable—things, with the postulates "known" properties of these realities, and that he considered the theorems additional facts about honest-to-goodness triangles and things. The accumulation of these geometric facts—from those developed by Euclid, Archimedes, and Apollonius to the proof of the constructibility of the regular 17-sided polygon by Gauss, including the 27 straight lines that lie wholly in the general cubic surface, and even the 369 proofs of the Pythagorean theorem so lovingly collected by Loomis—is all in some sense a part of geometry.

The method used in arriving at these new facts was the great distinctive contribution of the Greeks to human knowledge. It was the method which was then divorced from the subject matter of geometry and utilized not only in other branches of mathematics but in the sciences as well, and, indeed, even in theology.

Now, what are the implications of what we have been saying for the secondary schools? It is a legitimate concern and responsibility of the secondary schools to teach some of the knowledge that has been traditionally associated with the name geometry. It is a unique responsibility of the high school to give students at least their first contact with the unique method of geometry. This is exactly what we have been trying to do with one year or one and one-half years of demonstrative geometry. Then why has there been a continuing crisis in the teaching of geometry for fully 25 years?

This is another great and involved problem. I have no intention, now, of attempting to appraise all that has been said and written on the subject. I may, consequently, sound dogmatic when I merely state my own synthesis of the mountains of paper and oceans of ink that have been expended on it. We have undoubtedly set innumerable young mathematicians and scientists on their paths by teaching them geometry well. But I believe also that, by a slavish adherence to the Elements of Euclid and by ignoring the spirit in which he wrote his book (remember that the Elements were a comprehensive text of not only the geometry but also the algebra and the theory of numbers of his day), we have succeeded in doing three things: first, in driving away from geometry droves of pupils who couldn't take it; second, in boring great numbers in the middle-ability groups and giving them a distaste for mathematics; third, in wasting the time of the capable students.

The time is ripe and overripe for a solution of the problem. With the current concern over strengthening our teaching of mathematics and over doing something for the capable student, we have a propitious atmosphere in which to work out a genuine solution. One or two warnings, however, before we look at some proposals.

We may decide to return to the good old geometry—whatever we might mean by that. To do this would be to disregard all the valid criticisms that have been made of the teaching of geometry over the last two generations—and, especially, to ignore the claims of modern materials and needs in mathematics.

Another alternative is merely to speed up. The advanced standing program is attracting increasing numbers of capable high school students. Many schools, to enable their bright students to enter the program, are speeding up their preliminary mathematics, and often they cover the usual program of geometry in six months. The danger in this approach is that even the bright student may come through the experience with a blurred impression of what geometry is. The major facts are lost in a sea of unimportant detail and the logical structure is lost because the all-important details of logical thinking are slurred over.

We may, finally, decide to teach "pure" geometry, at least to the able student. Now, it is easy enough to say that geometry is an abstract science in which undefined terms (by the way, how do you *un*define a term?) and unproved statements are used to define additional terms and in which, then, we deduce statements which are called theorems. I have known young, enthusiastic, inexperienced teachers who taught all of demonstrative geometry in one lesson, just by saying this to a class. Of course, it is not that easy. It is precisely the process of leading the students to reach the stage at which this definition of geometry becomes meaningful that is the essence of teaching mathematics.

I hasten to interpolate here that this is not intended to be a criticism of the geometry programs at Illinois or at Muncie —two of the attempts at teaching pure geometry that are attracting attention. In both of these there is a genuine resolve to teach a rigorous geometry, and in both there is a built-in recognition of the fact that abstraction is something that happens after experience from which abstractions can be distilled. But how long does this take, and when do you reach the area of the circle?

What, then, is the way out? It must be found pragmatically by the teachers of mathematics, by the mathematicians, and by all those who have a stake in and a responsibility for education. It must be based on what the purposes of education are deemed to be, on what the needs of students are and are likely to be, on what the state of development of mathematics and geometry actually is, and on what the teachers of mathematics can do or are likely to learn to do.

I see greatest hope in the work that is being developed now by the Commission on Mathematics with its necessarily limited concern with the college-bound student, and in the plans of our own Secondary School Curriculum Committee with its broader objectives. The program proposed by the Commission and actually evolving in practice is one that points to a course in geometry with a drastically reduced content in synthetic geometry, a substantial introduction to co-ordinate geometry, and a treatment of some aspects of space geometry.

If a course in geometry follows a course in algebra that is not merely a bag of tricks but a genuine introduction into the concepts and methods of mathematics, the teacher of geometry can get over his obsession with the deductive method. The classical treatment of geometry can be illustrated by a sequence of theorems that reaches the Pythagorean theorem rather quickly. This material can also be utilized to give a concrete illustration of a formal

use of the deductive method. To be sure, the students would be treating geometry as a physical science and therefore only as a model of an abstract deductive system. This, however, along with similar experiences with algebraic and numerical materials, could be the first step toward a study of abstract deductive systems.

Parallelism—which would be in the deductive chain leading to the Pythagorean theorem—could now be used to set up a parallel network in the plane. By introducing appropriate additional assumptions (as, for example: "To every point in the plane of two real co-ordinate axes there corresponds one and only one ordered pair of real numbers") and by utilizing the Pythagorean theorem, we can establish the distance formula. Much of the traditional content of plane geometry—such as the properties of parallelograms, the areas of plane figures, and especially an enriched treatment of loci—can now be studied by the methods of co-ordinate geometry. Enough of this should be done to develop a facility in the use of the methods and an understanding of the concepts of co-ordinate geometry.

Space concepts and loci and the mensuration of solids—important in themselves and so necessary for a study of the calculus—cannot be left out of secondary school mathematics. There is no need, however, to take a half a year for these topics, nor need they be presented in the Euclidean strait jacket that characterizes most courses in solid geometry. They can be developed informally in connection with topics in plane geometry, and even by means of a three-dimensional co-ordinate system and the beginnings of the integral calculus. There is a further reason for teaching and even emphasizing the geometry of the sphere: it provides an easily accessible model of another, a non-Euclidean, geometry.

This program for geometry—which is substantially what the Commission recommends—is feasible for the college-bound, capable student of mathematics. The broader problem of *what* geometry, for whom, and how, should be one of the main concerns of our Curriculum Committee. By and large the same kind of thing is indicated there, too.

Many of the newer textbooks in geometry are helping to realize this program by including more and more newer materials: co-ordinate geometry, space concepts, spherical geometry, vectors. Under the leadership and inspiration of authoritative national bodies such as the Commission and the Curriculum Committee, they will also help in the other half of the program, that of laying the foundations of a richer and more rigorous treatment of the deductive method and of courageously leaving out a good part of the traditional content of synthetic geometry.

The ultimate worker of the miracle will have to be the teacher in the classroom. Overburdened as he already is, he has to be helped to find the time and the energy to refurbish his own mathematical preparation. He has to be given the opportunity of trying new materials and of making mistakes with them; remember that teaching anything for the first time is difficult. And he has to be encouraged to persevere in his noble and essential work.

Graphical construction of a circle tangent to two given lines and a given circle

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Geometry teachers will enjoy this proof of a familiar construction problem.

SEVERAL SOLUTIONS are available to us. One solution involves the construction of a circle passing through two points and tangent to a line,[1] which in turn requires the construction of the geometric mean of two segments.

Another solution involves inversions.[2]

A third solution[3] requires the measurement of the angle between the given lines and the determination of the cosine of half of it.

Here is a simple solution which requires the following theorem.

Assume a circle (O) has been drawn tangent to the two given lines AM and AN, and tangent to the given circle (O'). Draw tangents $A'M'$ and $A'N'$ to the given circle and parallel respectively to AM and AN.

Theorem. The point of tangency, T, of the two circles is the intersection point of AA' and the circle (O') See Figure 1.

Call the radii of the two circles r and r' respectively. The centers lie respectively on the bisectors of the angles A and A', and AN is $\parallel A'N'$ and AM is $\parallel A'M'$, hence $AO \parallel A'O'$.

The triangles OAB and $O'A'B'$, rectangular at B and B' respectively, are similar

$$\therefore OA:O'A'=OB:O'B'=r:r' \cdots \quad (1)$$

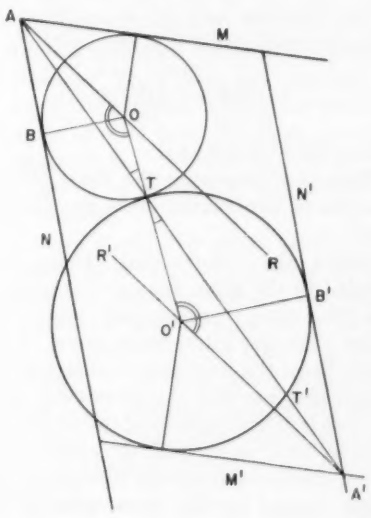
The center line OO' passes through the point of tengency T of the two circles. Connect T with A and A' respectively and consider the triangles AOT and $A'O'T$.

$$OT:O'T=r:r'$$

$$\angle ROT=\angle R'O'T$$

$$\therefore \angle AOT=\angle A'O'T.$$

Figure 1



[1] Nathan Altshiller-Court, *College Geometry* (New York: Barnes & Noble, 1950), p. 178.

[2] Jacques Hadamard, *Leçons de Géometrie Elémentaire I* (Paris: A. Colin, 1920), p. 227.

[3] Julian Lowell Coolidge, *A Treatise on the Circle and the Sphere* (Oxford: Clarendon Press, 1916), p. 185.

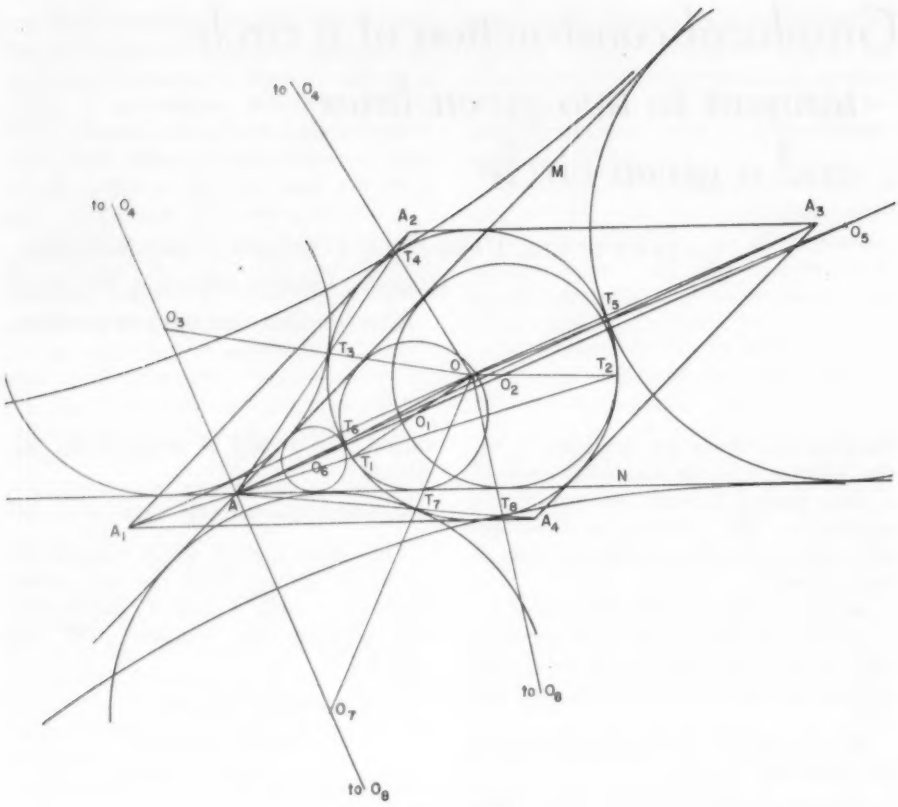


Figure 2

This together with (1) shows that the triangles AOT and $A'O'T$ are similar.

$$\therefore \angle OTA = \angle O'TA'.$$

Hence ATA' is one line.

Since AA' intersects the circle (O') in two points, two solutions now become possible.

Hence the construction: Draw lines parallel to the given lines and tangent to the given circle; connect their intersection point with the intersection point of the given lines; the intersection points of this connecting line with the given circle gives the two points of tangency; the connecting lines of these points with the center of the given circle intersect the bisector of the angle formed by the given lines in two points which are the centers of the required circles.

Now, four lines, parallel two and two to the given lines may be drawn; they form a parallelogram $(A_1A_2A_3A_4)$. A_1A, A_2A, A_3A, A_4A intersect the given circle in eight points of tangency, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8. T_1O, T_2O, T_3O, T_4O, etc., intersect the two bisectors of A in eight points-centers of the required circles (the bisectors of the obtuse angles corresponding to the vertexes of obtuse angles of the parallelogram). See Figure 2.

If one of the given lines does not intersect the given circle, only four solutions are possible.

If none of the given lines intersects the given circle, only two solutions are possible.

Solid geometry as an entrance requirement for engineering schools

M. L. KEEDY, *University of Maryland, College Park, Maryland.*

Some secondary schools are concerned about current recommendations to abandon the course *in solid geometry. What do engineering schools do about solid geometry as an entrance requirement?*

MANY SECONDARY SCHOOLS have dropped solid geometry as a full semester course from their curriculum. The Commission on Mathematics of the College Entrance Examination Board has recommended that the basic three-dimensional concepts should be taught to college preparatory students, but indicates that a full semester course is not essential. Curriculum planners may be reluctant to drop the traditional course, since there are certain engineering schools which require it for entrance. Therefore it is appropriate to ask which, and how many, engineering schools still list a semester of solid geometry in their entrance requirements. This article is a report of a study of this question.

There are 147 schools in the United States offering a first degree in engineering and accredited by the Engineers' Council for Professional Development. All of these were asked, by means of a short questionnaire, the following questions:

1. Do you now require one semester of high school solid geometry for entrance to the engineering curriculum?
2. If so, do you contemplate eliminating this requirement?
3. If so, about when?
4. If you have this requirement, is a student ever admitted without it?
5. If so, is he required to take a no-credit course in solid geometry after admission?

Replies were received from 134 of the 147 schools. There were 38 schools that indicated that they have the requirement and 96 that indicated that they do not. Of the 38 having the requirement, nine plan to eliminate it, and another 23 of them say that a student may enter without having had the course, but must take it after entrance. In making this count, schools which require a student to take a make-up course after entrance were included in the "required" category, even though those schools did not actually list solid geometry as an entrance requirement. Also included in the "required" category were two schools that do not now have the requirement but are contemplating it, and one school that returned incomplete information. Thus there are a maximum of six of the replying schools for which a semester of solid geometry is an aboslute requirement for entrance and is to remain so, or in which the requirement is being planned.

There is apparently great variation in admission procedures among those schools which admit students without the course but require it after admission. Remarks on some replies indicate that it is only occasionally that they admit a student without solid geometry, others say they very rarely do, and still others seem to indicate that many of their students enter without it.

TABLE 1

Required		Not required		No reply	Total
No definite plan to eliminate	Plan to eliminate	Recommended			
29	9	17	79	13	147

In several replies it was reported that some study is being given to the possibility of eliminating the requirement, but only those schools with rather definite plans are listed as planning to eliminate.

Of the 96 schools in the "not required" category, 17 list solid geometry as a *recommended* high school course. Counts in all categories are listed in Table 1.

Table 1 appears to show that those concerned with mathematics curricula need not be highly concerned about entrance requirements of engineering schools if they are contemplating the elimination of solid geometry as a semester course. That much is probably true. It would be a serious error, however, to infer that the engineering colleges are apathetic or unconcerned, or that they are lowering their entrance requirements. Remarks on the replies show the opposite. The dean of one school, for example, feels strongly that recommendations to include three-dimensional concepts at places in the curriculum other than the traditional course will result in "a simple deletion of solid geometry from the curriculum." Many schools are now substituting a semester of trigonometry for their former solid geometry requirement. Others require three and a half or four units of high school mathematics. Others state that they prefer more algebra in place of solid geometry, and some will accept other courses in lieu of it, such as an introduction to analytic geometry and calculus. One school in the "not required" category now requires analytic geometry for entrance, so that the freshman begins his college work in calculus. Several have cited their accept-

TABLE 2

SCHOOLS REQUIRING A SEMESTER OF SOLID GEOMETRY FOR ENTRANCE TO ENGINEERING CURRICULUM, OR IN WHICH IT MUST BE TAKEN AFTER ADMISSION

* University of Akron
* Alabama Polytechnic Institute
* University of Arizona
Brown University
** Bucknell University
**** The Citadel
** University of Hawaii
* University of Kentucky
University of Massachusetts
* Michigan College of Mining and Technology
* University of Minnesota
* Montana School of Mines
* Montana State College
* North Carolina State College
* North Dakota State College
*** University of Notre Dame
* Ohio University
* University of Oklahoma
* University of Rhode Island
* University of Tennessee
* University of Texas
* Texas Western College
* University of Tulsa
* Union College
* Virginia Military Institute
* Virginia Polytechnic Institute
* University of Virginia
* Wayne State University
West Virginia University

* Student can be admitted without course.
** Contemplates making course a requirement.
*** Requires solid trigonometry.
**** Insufficient information in reply.

TABLE 3

SCHOOLS WHICH REQUIRE SOLID GEOMETRY BUT PLAN TO ELIMINATE THE REQUIREMENT

University of Alaska (1959)
Case Institute of Technology (1959)
University of Cincinnati (1959)
University of Delaware (as high schools get new programs)
University of Louisville (1958)
Missouri School of Mines & Metallurgy (no date given)
South Dakota School of Mines (1958)
University of Toledo (1960)
University of Wichita (1959)

TABLE 4

SCHOOLS WHICH DID NOT REPLY

University of Alabama
Bradley University
University of California
Columbia University
Dartmouth College
University of Dayton
Duke University
Howard University
University of Michigan
Rose Polytechnic Institute
Syracuse University
United States Naval Postgraduate School
University of Vermont

ance of the recommendations of the Commission on Mathematics.

Disagreement on the value of high school solid geometry for engineers appeared in the replies. Some of the remarks indicated a strong feeling that the course is essential, but none of these replies referred to an objective study. On the other hand, all of the several replies that mentioned studies of student performance indicated that high school solid geometry makes little difference in the performance in engineering school. For example, one university reported "[Solid geometry as a requirement] was eliminated about 1935 after a study of our students indicated that those admitted without it did as well as those with it in subjects such as trigonometry, engineering drawing, descriptive geometry, and analytic geometry."

Schools which have the solid geometry requirement, as well as those schools for which replies were not received, are listed in Tables 2, 3, and 4.

This introductory paper . . . should be taken as a comment upon the remarkable and apparently universal human instinct for representing the most complex and diverse ideas in the form of *networks* or, as they are variously called, *flow-charts, topological graphs, trees, block-schematics, sociograms.*

There is scarcely a field of human thought which does not employ the concept of network: history, genealogy, administration, engineering, genetics, mathematics, sociology, neurology . . . the list is endless. Any concept so universal is worth examining, for it suggests that people in scattered disciplines have something in common. Of course the whole of mathematics is a common "language," universally applicable and being abstracted from particular diverse interpretations; but the network concept is one which is widely used also by those having little or no mathematical training. It is natural and instinctive.

* * *

Whenever we think about complex organizational tasks of any kind, the natural form of doodling is to draw networks—block diagrams, little flow charts . . . showing *relationships* between people, or things, or abstract ideas. This is so because the majority of us prefer to handle many concepts in pictorial form, rather than, say, algebraic or symbolic form; a network is the simplest way of representing relationships—it is the simplest language or "syntactical system."
—*E. Colin Cherry, Generalized Concept of Networks,* published in *Information Networks,* Symposium Proceedings, Polytechnic Institute of Brooklyn.

• HISTORICALLY SPEAKING,—

Edited by Howard Eves, University of Maine, Orono, Maine

Living costs a century ago

by Cecil B. Read, University of Wichita, Wichita, Kansas

The author of a textbook often finds it difficult to keep current with prices, wages, and the like. Students are quick to pick up discrepancies between values stated in problems in a textbook and actual conditions in life. Nevertheless, although there may be some lag, textbooks do offer an interesting commentary on prices of approximately the period in which they are published.

Some of the information might be of value when the student wishes to make certain comparisons. Interesting material may often be uncovered.

An arithmetic text of approximately a century ago was examined for some of the facts and prices which follow. The author of the text was Benjamin Greenleaf, and the title of the text is *Introduction to the National Arithmetic on the Inductive System, Combining the Analytic and Synthetic Methods; in which the Principles of the Science are Fully Explained and Illustrated. Designed for Common Schools and Academies.* (Quite obviously the author did not believe in brief titles.) This was the "new stereotype edition" published in Boston in 1858.

In the line of food prices, one finds problems involving milk at 5¢ a quart, raisins at 7¢ a pound, butter at 12¢ a pound, beef at 9¢ a pound, cheese at 9¢ per pound, coffee at 25¢ a pound. Not all prices were so appreciably lower than the present-day prices. For example, lemons are quoted at 88¢ per dozen, sugar at 8¢ a pound, tea at 62¢ per pound. Among other commodities, one finds a problem involving a price of $5 for a good pair of boots, $12 for a coat, $6 for a vest, and $73 for a watch. Corn is involved in problems at a price of 75¢ per bushel, wheat at 95¢ per bushel, coal at $10 per ton. Textbooks seem considerably less expensive; a problem mentions a student paying 25¢ for an arithmetic and 67¢ for a geography.

The student of the present generation who chooses to browse through such a book may be somewhat puzzled by firkins of butter, hogsheads of molasses, quintals of fish, nails of cloth. Likewise he would probably not have much basis for comparison with the prices for wagons, harness, chaises, or yokes of oxen. He might likewise wonder what was the content of "temperance wine."

However, before one concludes that it would have been a very fine time in which to live, it would be worthwhile considering some wages. The problems involving rates of pay are relatively few in this text, but we find such things as: carpenters working a day fifteen hours long, a clergyman's salary of $700 a year, a laborer being paid 37½¢ per day, another laborer's wages being quoted at $7 per week, farm labor at $3 per month and board, and plastering being done for 10¢ per square yard.

It might well be that the student who has access to a textbook of this date might find not only considerable interest in the problems of the time, but might find a very interesting study in relative costs of living.

Benjamin Franklin and mathematics

by Richard W. Feldmann, Jr., University of Buffalo, Buffalo, New York

Although Benjamin Franklin (1706–1790) was one of America's foremost inventors and natural philosophers, he cannot be classified as a great mathematician. His belief that the only value of science is its practical side is shown by his statement, "Nor is it of much importance to us to know the manner in which nature executes her laws. 'Tis enough if we know the laws themselves. 'Tis of real use to know that china left in the air unsupported will fall and break, but how it comes to fall, and why it breaks, are matters of speculation. 'Tis a pleasure indeed to know them, but we can preserve our china without it."[1]

When nine years old, according to his *Autobiography*, Franklin was enrolled in Mr. George Brownell's school for writing and arithmetic, where he "failed in the arithmetic, and made no progress in it."[2] While he was serving his apprenticeship in his brother's printing shop, Franklin read Cocker's book on arithmetic and two works on navigational geometry "with great ease."[3]

The only published article on mathematics by Dr. Franklin appeared in the October 30, 1735 issue of his *Pennsylvania Gazette*. In *On the Usefulness of the Mathematics*,[4] the practical aspect is stressed. Arithmetic is said to be for merchants, shopkeepers, and traders; geometry is for architects, astronomers, geographers, mariners, and surveyors. The article concludes with a sketch of mathematics in ancient history.

In an attempt to establish an educational academy in Philadelphia in 1749, Dr. Franklin published his *Proposals Relating to the Education of Youth in Pensilvania*, in which he suggests a curriculum for the academy. The only mathematics included are "arithmetick, accounts, and some of the first principles of geometry and astronomy."[5] Accounting is stressed in a footnote with the comment that the keeping of accounts is essential for all gentlemen.

The remainder of his mathematical efforts are found in his letters to friends. The only two in his collected works were both written to Peter Collinson of England about 1750. The first[6] of these concerned magic squares, that is, square arrays of numbers such that the sums of the numbers in each row, in each column, and in each of the two principal diagonals are identical. In an n by n square, the elements are the integers from 1 through n^2. In Franklin's letter there are two magic squares, an 8 by 8 and a 16 by 16. Unfortunately, according to Albert Chandler in the *Journal of the Franklin Institute*,[7] the 16 by 16 square was set up wrong by a printer in one of the early editions of Franklin's *Works*, and this incorrect square turns up in the later editions. Mr. Chandler provides a corrected square which is reproduced in Figure 1. Dr. Franklin referred to this one as the "most

[1] Benjamin Franklin, *Works, with a Life and Notes by Jared Sparks* (London: published in ten volumes by Benjamin Franklin Stevens, 1882), vol. 5, p. 234. Hereafter this work will be referred to as "Franklin, *Works*."

[2] Benjamin Franklin, *Autobiography, with an Introduction by Woodrow Wilson* (New York: Century, 1901), p. 13.

[3] Ibid., p. 26.

[4] Franklin, *Works*, vol. 2, pp. 66–70.

[5] Benjamin Franklin, *Proposals Relating to the Education of Youth in Pensilvania* (Philadelphia: 1749), p. 14. Facsimile Edition by William L. Clements Library, Ann Arbor, 1927.

[6] Franklin, *Works*, vol. 6, pp. 100–103.

[7] Albert Chandler, "Benjamin Franklin's 'Magic Square of 16,'" *Journal of the Franklin Institute*, 251 (1951), 415–422.

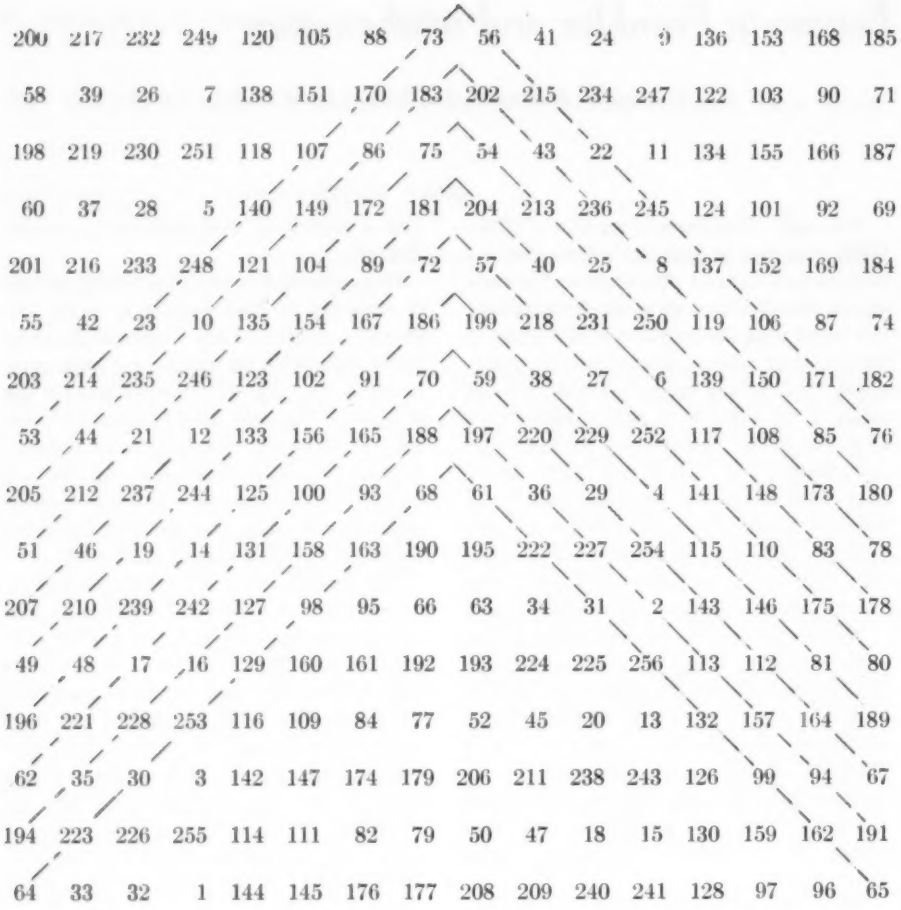


Figure 1
The 16-by-16 Magic Square

magically magical of any magic square ever made by any magician,"[8] because it has additional properties. The bent diagonals (indicated in Fig. 1) also total the same, namely 2056, as the rows, columns, and principal diagonals. Bent diagonals also run from top to bottom, left to right, and right to left, but, for simplicity, are not shown. The elements in any 4-by-4 subsquare also add up to 2056. Although proud of his squares, he speaks of the time consumed in constructing them as time "which I still think I might have employed more usefully."[9]

In the second letter,[10] Dr. Franklin produced a magic circle which is shown in Figure 2. Its magical properties are that the numbers in any radial row, added to the central 12, total 360, the number of degrees in the circle. The halves of the

[8] Franklin, *Works*, vol. 6, p. 103.

[9] *Ibid.*, p. 101.
[10] *Ibid.*, pp. 104–105.

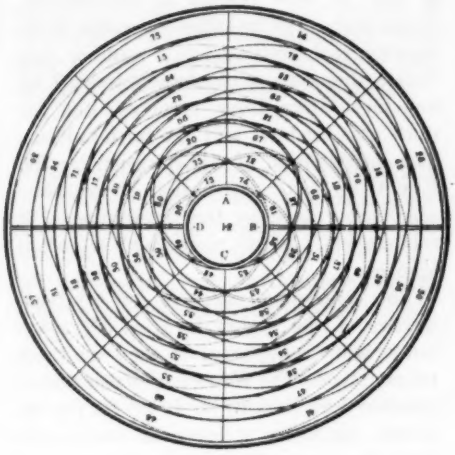


Figure 2
The Magic Circle

numbers in a radial row, when added to 6, total 180, the number of degrees in half a circle. The halves of the numbers in the concentric semicircles above or below the double line also add up to 180 when added to half the central number. Any four adjoining numbers (making almost a square) plus 6 total 180. There is an additional property related to the twenty eccentric circles in that the elements in each of these, when added to the central number, total 360.

An indication that Benjamin Franklin's mathematical abilities were intuitive is his statement that the population of the United States will double itself every twenty-five years. He seems to have made this assumption with almost no data as a guide. As an example, the 1790 census recorded 3,929,214 people. By calculation the 1890 census should have been 62,867,424. By actual count, it proved to be 62,947,714, an error of only 0.13%.[11]

Although Benjamin Franklin perhaps possessed the ability to become one of our country's great mathematicians, his desire for practical inventions and discoveries, as well as his duties as a printer, leader, and statesman, prevented him from reaching this status.

[11] Conway Zirkle, "Benjamin Franklin, Thomas Malthus, and the United States Census," *Isis*, 48 (1957), 58–62.

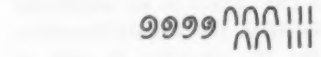
Note on Egyptian numeration

by Carl B. Boyer, Brooklyn College, Brooklyn, New York

Many histories of mathematics give an erroneous impression of the primitivity of Egyptian notations for whole numbers. In these textbooks it is either stated or implied that there was but a single scheme in use in Egypt—the so-called hieroglyphic notation. In this system the digits from one to nine were denoted by the appropriate number of vertical strokes, so that the integer six, for example, would be written

Multiples of ten were represented by repeated croquet wickets, so that fifty would appear as

Using the same principle, the number 456 was represented hieroglyphically as

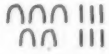


and, by the time of the great pyramid builders, numbers in the hundreds of thousands appeared in this notation. Numerical inscriptions on monuments continued to follow this pattern, much as Roman numerals appear on many of our tombstones in the twentieth century. But to claim that this is the manner in which Egyptians wrote their numbers on papyrus is tantamount to arguing that we today use Roman numbers in computations on paper. In particular, to leave the student of ancient mathematics with the impression that the Ahmes and Moscow papyri make use of the above-described system of numeration is to conceal the truth. These two papyri make use of a radically different notation, and one which is considerably more sophisticated than the hieroglyphic.

The Egyptians early recognized that the ancient hieroglyphic script could be considerably abbreviated through the use of conventionalized symbols, and the result was a more cursive script known as hieratic. This paleographic change resulted in a new basis for numeration in which a collection of symbols, such as

was replaced by a single mark or cipher, in this case

The number



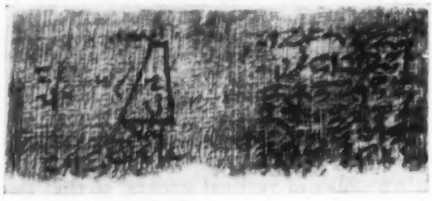
similarly came to be simply

This scheme necessitates a distinctive mark or cipher for each of the first nine natural numbers and for each of the first nine integral multiples of integral powers of ten. In this system of numeration, which might well be called a decimal cash-register cipherization, any number less than a thousand would be represented by not more than three symbols—one for units, one for tens, and one for hundreds. The introduction of cipherization by the Egyptians is a contribution of considerable importance to the development of numeration, and one which is all too often ignored.

The widespread neglect by textbook writers of the higher form of Egyptian numeration is apparent in even the most recent books. The *History of Mathematics*, published in 1958 by J. F. Scott, for example, continues to leave the impression that the numeration in the Ahmes and Moscow papyri is hieroglyphic, although a glance at the reproduction of a portion of the Ahmes papyrus given in Plate I of the book will show that this is not the case.

That the hieratic numeration does indeed represent a distinctively new scheme is confirmed by the fact that in it the digits in ascending order are written from left to right, whereas in the hieroglyphic notation the sense is reversed. The total dissimilarity of hieroglyphic and hieratic numeration is brought out clearly by a glance at the celebrated problem in the Moscow papyrus in which the volume of a frustum of a square pyramid is found. Just inside the trapezoidal diagram (representing the frustum) are two numbers: the upper single mark is the cipher for 6; the lower two marks represent 56, but the left-

Figure 1



hand mark is the cipher for 6 and the right-hand mark is the cipher for 5. On the extreme left-hand side of the portion of the papyrus in our illustration one finds near the top, and again in the lower left-hand corner, the two-digit number 28,

written with digits reversed. In the middle on the left-hand side are three marks, the first two of which again represent the number 56 (compare with the lower number inside the figure). Just above the figure one finds the numbers 2 and 4, written as repeated vertical strokes; and below the figure is another set of four strokes. To the right of the figure are the instructions for computing the volume of the frustum, knowing that the upper and lower bases are squares of sides two and four, and that the altitude is 6. One may descry again in the instructions the ciphers for 6 and 28 and 56, among others; but one will find none of the conventionally described hieroglyphic numerals beyond the one for 4. One cannot avoid the conclusion that many textbook accounts of Egyptian numeration are inadequate.

Have you read?

Pack, Elmo. "Mathematics and Science," *The Clearing House*, September 1958, pp. 13–14.

Here is an interesting article on the integration of mathematics and science in the junior high school. The author believes that only the outdated parts of these subjects are not integrated. He goes on to show that historically they developed as one, that proofs and answers in both are a result of the deductive process, that they both use the same language and involve the same reading difficulties and skills, and that the vocabularies overlap.

The student who has a command only of one or the other misses the major concepts of living, the exercises he works are not real, and—most serious of all—he does not acquire the psychological mind-set for real and lasting learning in either science or mathematics. Read this article and try out the author's theories.—Philip Peak, *Indiana University, Bloomington, Indiana.*

Rosenbaum, E. P. "The Teaching of Elementary Mathematics," *Scientific American*, May 1958, pp. 64–73.

Many of you are wondering what may happen to the teaching of mathematics in the secondary schools during the next decade. This article offers some possible answers. One of the prominent ideas is that of a "set" or "collection of things"; another is the notion of operations with the set. This article gives several illustrations as to how sets aid in the study of inequalities, in the understanding of numbers, in adding meaning to variables, and in clarifying solutions.

Geometry is probably due for a most drastic change in both the length of time devoted to it and the methods used in its study. The Illinois program is discussed, as are the College Board Commission recommendations. You also will want to study the illustrations.

This article can serve as a springboard into a further study of the growing and changing field of mathematics. Don't miss it.—Philip Peak, *Indiana University, Bloomington, Indiana.*

Rothman, Philip. "Who Is to Blame?" *Think*, March 1958, p. 28.

You are often subjected to questions and inferences about science education today not being up to par. In this short article the author suggests we ask ourselves as citizens of the community just what support we are providing for our schools.

For example, do we provide classes that are teachable in both size and number? Do we give teachers opportunities to keep up to date on their fields? Do we give teachers help to relieve them of clerical and police tasks, and accord them the respect due their profession? Have these teachers been selected on the basis of their qualifications, left free to teach with materials and books as they deem best, and given pay sufficient to attract and hold the best?

If we as citizens cannot answer yes to these questions, then don't place the blame for any deficiency on our "educationist" teachers or pupils. The means must be provided first. Recommend this article to all your friends in the community.—Philip Peak, *Indiana University, Bloomington, Indiana.*

Edited by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

Panel games

by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

A unique way to organize a classroom recitation period or to review a unit is to set up these activities as panel games.

What's My Line?

This game is played like a quiz contest or charades. The "line" to be guessed may be a famous mathematician, a mathematical word, number, or theorem. Whatever is to be guessed is described by a series of statements beginning with obscure and minute information and ending with a well-known description. The object of the game is to identify quickly the "line." The more information required for identification, the lower the score. For example, if the answer is guessed on the first clue, the player gets 10 points. If two clues are needed, 8 points are allowed, and so on. Groups may compete by making up clues and taking turns competing for points.

Here are examples of "lines" to guess, with appropriate clues:

1. Fraction
   a. I am usually a small number. (10 points)
   b. My appearance may be changed by division without changing my value. (8 points)
   c. Sometimes I am less than one. (6 points)
   d. Sometimes I am called a ratio. (4 points)
   e. Sometimes I am a per cent. (2 points)
   f. Sometimes I am a decimal. (1 points)
2. Rectangle
   a. I am a geometric figure.
   b. I am present in every room.
   c. Sometimes I am a square.
   d. My opposite sides are parallel.
   e. My area is found by multiplying my length by my width.
   f. My four corners are right angles.
3. Formula
   a. I am a way of expressing relationships.
   b. I am used to make predictions.
   c. I can be translated into a sentence.
   d. I can be changed in appearance by the multiplication axiom without changing in value.
   e. I am made up of symbols, variables, and numbers.
   f. I am an expression of equality.
4. Negative Number
   a. I am a certain kind of number.
   b. There are an infinite number of me.
   c. I am used to show direction as well as distance.
   d. I may be a decimal, fraction, or whole number.
   e. My value is always less than one.
5. Euclid
   a. I wrote a famous mathematical book.
   b. I lived about 300 B.C.
   c. My book is a best seller.
   d. My book was written in Greek but has been translated into most languages.
   e. My book is about geometric theorems.
6. Point
   a. I am a location in space.
   b. I have no dimensions.
   c. I am usually represented by a small black circle.
   d. I am the intersection of two lines.
   e. Two of me determine a straight line.
7. Coefficient
   a. I am part of an algebraic expression.
   b. I may be an arithmetic number or an unknown number.
   c. I am a multiplier.
   d. I have a long name that begins with C.
8. Per cent
   a. I am used to advertise a sale.
   b. I am used very often in a bank.

c. I am really a fraction.
d. I am changed to a decimal before I am used.

9. Angle
   a. I have no width or length but still can be measured.
   b. I am a simple geometric figure.
   c. I have only two sides.
   d. I have one vertex.
   e. I am measured in degrees.
10. Exponent
   a. In multiplication problems I am added.
   b. Sometimes I am called a logarithm.
   c. I am a short way of showing multiplication.
   d. I am expressed by a number written small.
   e. I tell how many times a number is taken as a factor.
11. Variable
   a. I am found in all formulas.
   b. I am usually represented by a letter.
   c. I usually have different values.
   d. I am used to represent unknown numbers.
   e. My name means changing.
12. Vector
   a. I am used to solve navigation problems.
   b. I am usually a signed number.
   c. I represent a distance
   d. I have size and direction.
   e. I am used to show the addition of forces.
13. Parabola
   a. I am the locus of points equidistant from a given point and a given line.
   b. I am a conic section.
   c. I am a curve that shows how a ball travels when it is thrown into the air.
   d. I am the graph of the equation $Y = X^2$.
14. Zero
   a. Some do not consider me a number.
   b. I am the last of our ten digits to be invented.
   c. I am often used to locate the decimal point.
   d. I can not be used as a divisor.
   e. Sometimes I am called a cipher.

Stump the experts

The class selects three to six persons to form a panel of experts. The class is then divided into two teams. Each team makes up a set of questions, problems, or terms to ask the experts. Members of each team then take turns asking questions. Whenever the experts answer a question correctly they get a point. Whenever the experts cannot answer a question, problem, definition, or proof, the team asking the question gets a point. Whenever the opposing team can answer a question that stumps the experts, it gets a point.

Ten questions

This is a guessing game similar to Twenty Questions. The class may be divided into teams or play the game as a group. A player is chosen as "Questioner" and stands with his back to the blackboard. A word is written above his head by the teacher or a member of the opposing team so that the class can see it but the "Questioner" cannot see it. The player who is the "Questioner" then tries to discover what the word is by asking the members of the class questions about it. Points may be awarded according to the number of questions needed to identify the word. If the "Questioner" needs only one question, he gets 10 points, two questions, 9 points, and so on. Questions are to be answered only with a *yes* or *no*.

As a variation of this game, use theorems, formulas, numbers, problems, or mathematicians instead of problems.

Here are sample items for Ten Questions.

Words: per cent; measure; equation; binomial.

Formulas: $A = lw$; $I = PRT$; $S = \frac{1}{2}gt^2$; $a^2 + b^2 = c^2$.

Numbers: 5; II; $\sqrt{3}$; $2i\pi$.

Mathematicians: Einstein; Euclid; Newton; Archimedes.

Theorems: Pythagorean theorem; the sum of the angles in a triangle; the locus of points equidistant from two points; the opposite sides of a parallelogram are equal.

Problems: (1) Find the number if 12% of it is 24. (2) What is Mark's age if four less than twice his age is 38? (3) What is the distance a 26-inch bicycle wheel travels in one revolution? (4) Find two locker numbers whose sum is 50 and whose difference is 20.

Arithmetic warm-up

This is a take-off of the old fashioned spelldown. Instead of spelling words, the participants work problems mentally. Teams are chosen from different classes to compete with each other at a school or a community function. In one city, a team of sixth-graders has competed successfully with a team of adults at a PTA meeting.

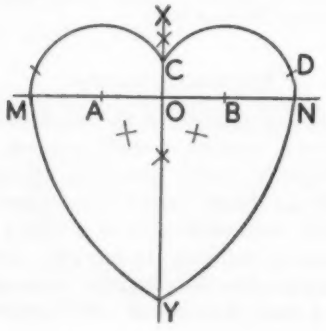
Problems are selected which emphasize understandings or basic ideas rather than rote memorization. Beginning competition can be somewhat softened by eliminating players only when they have missed two problems. Players drop out only when the missed problem is worked successfully by a member of the opposing team. Problems are given to the opposing teams in alternating order. The problems used should be selected according to the grade level involved.

A heart for Valentine's Day

by Mae Howell Keiber, West High School, Waterloo, Iowa

What could be more appropriate for the mathematics class than a heart construction for Valentine's Day—especially if the construction is geometric? The valentine heart is a cardioid, but its construction can be simplified to involve the constructions of plane geometry. Last year my classes followed the directions below on February 13 and were pleasantly surprised to end with a valentine. This inspired a variety of geometric valentines with appropriate and clever verses. This year we are making hearts out of red construction paper and paper doilies and sending the best ones to our hospitals. You'll be pleased with the response to this seasonal variation in your regular class routine.

These are the valentine construction directions:



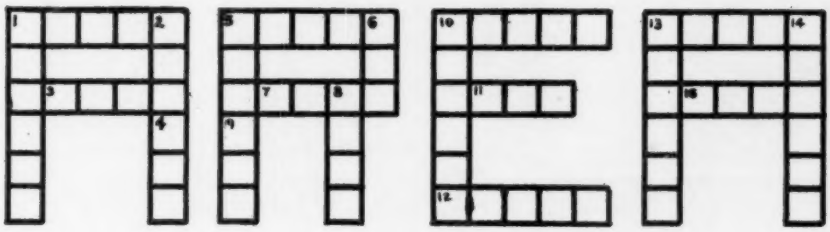
1. Draw a segment AB equal to $1\frac{1}{2}$ inches in the center of your paper.
2. Construct a perpendicular bisector of AB intersecting at O. Call it XY.
3. Measure OY equal to $2\frac{1}{2}$ inches.
4. With A and B as centers and a radius of $\frac{3}{4}$ of an inch, construct arcs of circles intersecting XO at C and AB extended at points M and N respectively.
5. Construct a perpendicular bisector of the distance MY and let it intersect the arc CN at D.
6. Using DY as a radius and D as a center, construct an arc from M to Y.
7. In like manner, construct an arc from N to Y, using a point on arc MC as a center.
8. When the figure is completed, write an appropriate saying on it.

Review tests can be different

by Louise Hazzard, State College Junior High School, State College, Pennsylvania

At the beginning of the school year we frequently give achievement tests and find that there are certain sections of the work of the previous year which the students have forgotten over the summer vacation. When the achievement tests are discussed carefully with each student we can then find his weak points and provide remedial instruction. A brief discussion of some of the work, and a few tests of the type illustrated below, are often, during the first two months of school, helpful and encouraging to the student in his recall of preceding work. This type of test is also interesting when it is given at the end of a chapter or unit in the regular mathematics class. I use it in my seventh and eighth grade classes for both of these purposes and get an enthusiastic response from my students.

A sample review test on area follows.



ACROSS

1. Find the area of a rectangle 212 ft. long and 132 ft. wide.
3. Find the area of a triangle if the base is 94 ft. and the height 92 ft.
5. Find the area of a rectangle when the height is 72 ft. and the base is twice the height.
7. What will be the area of a circle whose radius squared equals 721? ($\pi = 22/7$)
10. What is the area of a rectangular lot 300 ft. by 320 ft.?
11. The dimensions of a lot are 75 ft. by 42 ft. A walk 3 ft. wide is to be made around the rectangular lot. What area has the walk?
12. What is the area of 2 lots shaped like trapezoids if each has a height of 100 ft.? One has an upper base of 100 ft. and lower base of 110 ft.; the second a lower base of 120 ft.
13. Find the area of a rectangular figure 251 ft. long and 201 ft. wide.
15. Find the area of a parallelogram having a base of 74 ft. and a height of 58 ft.

DOWN

1. Find the area of a circle whose diameter is 6 inches. ($\pi = 3.1416$)
2. Find the area of a square with each side equal to 22 in.
4. Find the circumference of a circle whose radius is 21 ft.
5. Find the area of a trapezoid with the following dimensions: height, 10 ft.; upper base, 11 ft.; lower base, 15 ft.
6. What is the area of a parallelogram with a base of 47 in. and a height of 18 in.?
8. Find the area of a square when one side equals 81 ft.
9. What is the area of the surface of a cylinder whose radius is 7 in. and whose height is 5 in.? ($\pi = 22/7$)
10. What is the square of 300?
13. What is the area of a sphere if the diameter is 40 in.? ($\pi = 3.1416$)
14. What is the area of a square that measures 320 ft. on a side?

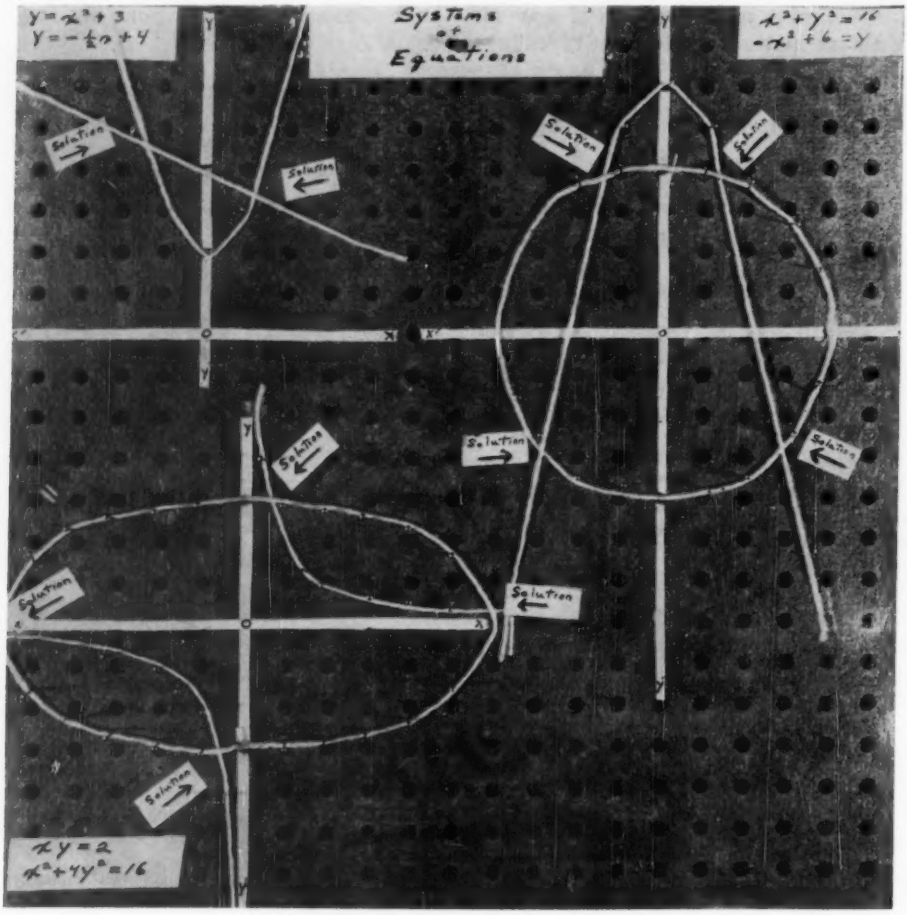
Using celotex for graphing

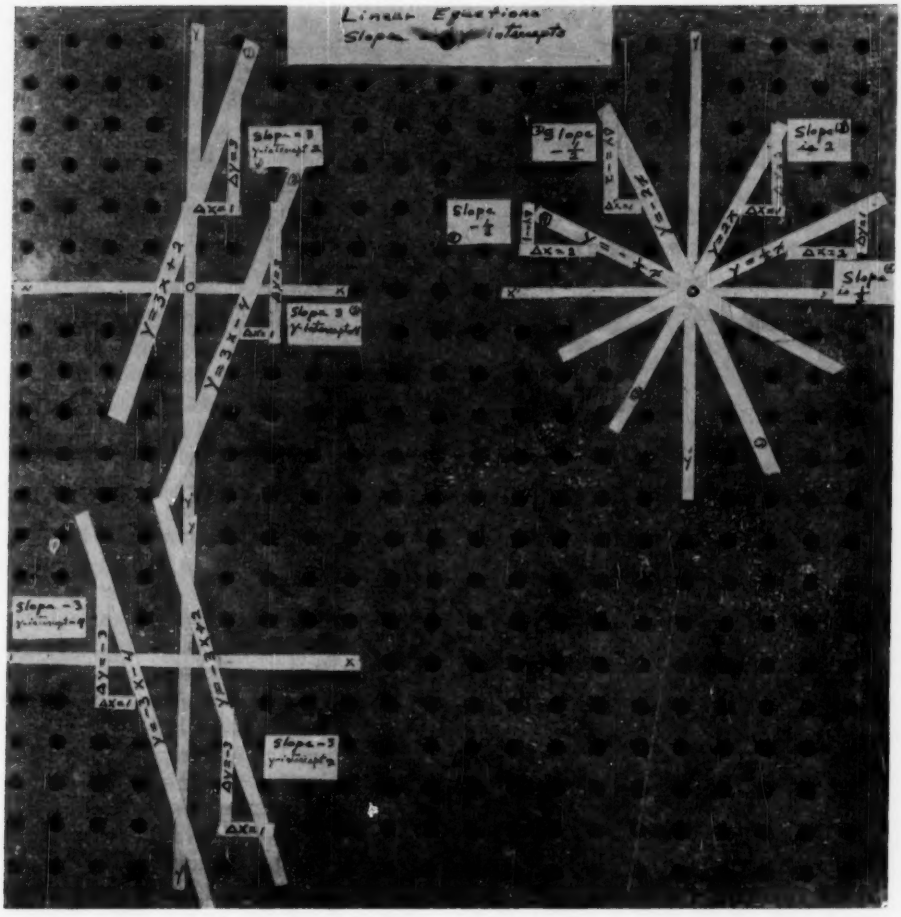
by Margaret Reinhold, Wilmington High School, Claymont, Delaware

This past year, an intermediate algebra class at Wilmington High School had an interesting and valuable experience in using acoustical celotex boards for graphing.

Each student had a celotex board 12″ by 12″, pins, match sticks, tooth picks or tacks, string, wire, colored yarn, colored strips of paper, adhesive tape, cellophane or lined tape, and colored pencils.

The unit consisted of drawing the graphs on the celotex board. Location of points and meaning of coordinates and the graphing of linear functions, with emphasis on slope intercept method, were reviewed. Match sticks were used for locating points





and red strips of paper for lines. The class also graphed quadratic equations. The class suggested that each student make a board of his own choice for an exhibit for the Parent's Party. Interest and motivation were high. Some students made several models, including cubic equations. Each student described his graph to the class.

Some of the other uses made of the boards are as follows:

1. To teach addition and subtraction of positive and negative numbers.
2. To illustrate trigonometric ratios.
3. To illustrate Pythagorean theorem.
4. To find areas.
5. To make a model of a cubic foot and a square yard.
6. To compare areas of figures with same perimeter.
7. To illustrate graphs of formulas.

Several pictures may help to illustrate the project carried on by the class.

A column of unofficial comment

Should we challenge elementary school mathematics?

by E. Glenadine Gibb, Iowa State Teachers College, Cedar Falls, Iowa

Present-day concern with regard to the secondary school mathematics program has resulted in efforts to give new direction to the development of mathematical concepts, to the organization of ideas, and to the method of teaching secondary school students. Mathematicians have urged that certain basic principles be developed in our schools. Furthermore, recent studies in the area of learning have implications for different and improved instructional procedures whereby we may provide for more effective learning experiences.

Inevitably come the questions: Is it possible that boys and girls might come into junior high school with a better background than they now have? Are we carefully planting the seeds of mathematical ideas in the elementary school, or are we scattering them in such a way that we promote careless growth of ideas, ideas which later must be greatly pruned, replanted, and even in some cases uprooted? Are we making the most of instruction in mathematics in the elementary school, or do we need a change of content and/or emphasis in the arithmetic of the elementary school?

Mathematics is both a science and a language. It involves order, patterns, and logical processes. It concerns relation, interaction, movement. Yet, as we study the arithmetic program of the elementary school, we find it possible for children to deal with ideas in isolation in such a way that results may have as little in common with mathematics, as understood by mathematicians, as spelling has in common with literature. Acknowledging present emphasis on meanings and understandings, have we not regarded mathematical ideas superficially? We have been concerned with *experiences which bring meaning to symbolism*, rather than the *symbolizing of mathematical ideas*.

The learning of mathematics is concerned with the creation of mental patterns, patterns such as an ordered sequence of number. Organization, the perception of relationships, comparisons, are essential in the development of mathematical ideas and form the basic media for conceptual learning. For a young child, the arrangement and rearrangement of objects in the classroom is substance mathematics. He is not aware of the significance but his teacher should be. Observations of 2 hands, 2 feet, 2 eyes, 2 ears, 2 balloons, 2 cars, 2 houses, 2 people lead to the idea that these things have "twoness" in common. Two becomes an independent idea. The relating of the written symbol to this idea is a further development—not one which develops along with it. Nevertheless, often in pres-

ent programs we do find this to be a simultaneous development. Many pictures in early number books, as well as many experiences in the classroom with collections of manipulative objects, emphasize the single-unit idea—4 cars, 6 eggs, 5 ducks. The entity of 4, perhaps as 4 cars on a transport and as 4 legs on a chair, or of 6 as 6 eggs in a nest, or of 5 as 5 fingers on the hand, is neglected. As stated by the Polkinghornes, "Placing a definite number of objects within a boundary enables children to appreciate better the entity of number and to analyze it."[1]

Mathematical ideas grow very slowly. As adults we tend to forget this, for what seems obvious to us we feel should be obvious to children. This does not necessarily follow. In the words of a child to his teacher, "This may be easy for you, but it's sure hard for us!" The process of abstraction is in itself complex. It involves the power to analyze a given situation or experience, to select and distinguish between relevant and irrelevant data, and the ability to classify and rearrange the data so that what is known to be significant may be abstfacted and finally symbolized for future application. In addition, there is a verbal factor which extends beyond mere facility in correct use of words, and there is the ability to manipulate concepts and ideas without the help of concrete aids. Piaget, in his study of mathematical ideas which children have, claims to have shown that number is not an innate idea but develops when children perceive the correspondence between equal sets.[2] He also contends that operation with number can take place only when the *ordinal* and *cardinal* aspects of number can be expressed in terms of each other.

From Mitchell's study of the nature of mathematical thinking, we find:

> A better understanding of the mental processes fundamental to mathematics finds expression in what is known as more "insightful" methods of teaching whereby the processes of education and abstraction would be stressed rather than that of deduction.[3]

Furthermore, in Russell's study of children's thinking, it is stated:

> The process of inductive thinking, which involves a search for common elements in a given group, may easily go astray because the process of generalization is neither understood nor consciously controlled by the child. But even more, a wrong generalization may easily be accepted by the child because he has no related experience by which he can check his conclusions. Thinking can be accurate only after a child has had much experience by which he can check his generalization or concept. Concepts can rarely embrace the total field involved and set and suggestibility may further limit the selection of items and the accuracy and usefulness of the concept.[4]

Realizing how slowly ideas develop, should we not give serious thought to the mathematics of the elementary school and the learning experiences provided? It would seem worthwhile to give careful study to our present arithmetic program in light of mathematical ideas which may better be developed or could be developed. We need experiences in our arithmetic classes which will enable children to begin forming mathematical ideas of number groups, quantity and space, and experiences which will lend themselves to the habits of mathematical thinking: order, a whole and its parts, and the combination of wholes to make another whole. In so doing, let us keep clearly in mind the critical points of development of mathematical understanding.

[1] R. K. and M. I. R. Polkinghorne, *Fundamental Number Teaching* (London: George G. Harrap and Company, Ltd.), p. 19.

[2] J. Piaget, *The Child's Conception of Number* (London: Routledge and Kegan Paul, 1952).

[3] F. M. Mitchell, "The Nature of Mathematical Thinking" (Doctoral dissertation, London, 1937, as reviewed in *The Teaching of Mathematics in Primary Schools*, a report prepared for the Mathematics Association [London: G. Bell and Sons, 1956]), p. 110.

[4] D. Russell, *Children's Thinking* (Boston: Ginn and Co., 1956), pp. 246–247.

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

College Plane Geometry, Edwin M. Hemmerling (New York: John Wiley and Sons, Inc., 1958). Cloth, ix+310 pp., $4.95.

Editorial note: this is a second review of this book. The first review appeared in the October, 1958, issue.

Topics that usually have appeared in traditional courses at the high school level are here presented in a direct and efficient manner. In addition to the usual material on polygons and circles, Chapters X and XI introduce trigonometry and solid geometry, respectively.

The text was written, according to the author, as a one-semester course for the "student more mature than is usually found among high school sophomores." The text would, of course, be useful for those college students who for some reason did not take geometry in high school. It is tacitly assumed, however, that (1) students of this type exist and (2) colleges would supply a teacher for such a course. The text would serve very well as a background course for technical courses or for further work in mathematics.

Because of the present curricular trend to revise the work in tenth grade geometry, this text could be very useful at this level. By using this text some time could be saved for the use of analytic methods as well as for an introduction to other geometries.

Another major objective for the text as given by its author is "to relate the abstract materials of geometry to experiences in the students' daily life." Whether this attempt to transfer geometric methods to other situations is a help or a hindrance to the usefulness of the text should be left to an objective evaluation of its users.

Since the text is, in general, an efficient version of the usual standard high school text in plane geometry, it follows that it is subject to any of the usual criticism leveled at these texts.

A fair evaluation of the text must be made in light of the present increasing emphasis on mathematics and the present upheaval in the mathematics curriculum. The author and the publisher should be complimented because the approach used in this text is an experiment in a very important area of mathematics.—*D. R. Bey, Illinois State Normal University, Normal, Illinois.*

An Emerging Program of Secondary School Mathematics, Max Beberman (Cambridge: Harvard University Press, 1958). Cloth, 44 pp., $1.50.

This book presents the Inglis Lecture for 1958 given at Harvard University by Max Beberman. The National Council of Teachers of Mathematics should be proud that one of its members was chosen for this honor.

In his lecture, Beberman explains how the University of Illinois Committee on School Mathematics was started and the nature of its work. He then explains in considerable detail how UICSM has tried to build a program of meaningful mathematics by emphasizing (1) *precision of language* and (2) *discovery*. He provides many examples so that communication is assured.

The reviewer believes that Beberman has described very well the most important principles under which UICSM operates. It is significant that Beberman does not mention "modern mathematics" in the lecture. This fact can be interpreted as meaning that the elements of "modern mathematics" in the UICSM program are not considered as important as is their approach of using precise language and discovery. This point of view is commendable.—*Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa.*

A Modern Approach to Intermediate Algebra Henry A. Patin (New York: G. P. Putman's Sons, 1958). Cloth, viii+224 pp., $3.75.

This book covers the standard material of the one-semester intermediate algebra course in the standard fashion. There are forty-eight "word problems" in the book; other exercises are drill problems in elementary manipulations. The techniques are clearly explained and the illustrative examples are good.

The "modern approach" is not so very modern. It consists of an attempt to state explicitly the assumptions about numbers that will be used, with proofs of some properties from these assumptions. The author disavows complete rigor, which is perhaps sensible. We can ask, however, whether the structure of algebra is clarified by the treatment.

There are two rigorous approaches to the number system, one constructive, the other axiomatic. One can begin with the whole numbers as given, for example, by the Peano postulates, and *define* in succession addition, multiplication, zero, negatives, rationals, reals, and complex numbers: a long, technical program. (Statements such as $ab=ba$ are then theorems, reproved at each new stage.) Or one can begin by *assuming* a field along with whatever additional assumptions are needed to narrow down to the real or the complex numbers. This book attempts to combine the two approaches. To give a sample, Patin takes the natural numbers,

1, 2, 3, · · · , as given, though addition and multiplication are undefined except by various postulates. Zero is obtained by Postulate 1.7, "There exists a unique number, 0, such that for any number a, $a+0=a$." The next statement is, "We can now prove that the product of any number with 0 is 0," followed by a "proof." Of course, if all one knows about 0 is 1.7, such a proof is impossible, and indeed statements assumed about the *positive* integers are used in the argument as though they applied to 0. The rest of the treatment is similar, with similar objections. It is hard to evaluate. On the one hand, the student will learn that $a \cdot 0 = 0$ is important; on the other hand, he cannot get a *clear* idea of why it is so from the book.

The traditional intermediate algebra is full of things other than foundations that many mathematicians would like to see clarified. Not many of these are improved here.

Though by no means an adequate reply to the demand for curricular reform, the book is probably quite teachable and is a step forward. —*G. S. Young, University of Michigan, Ann Arbor, Michigan.*

Plane Geometry, Royal A. Avery and William C. Stone (Boston: Allyn and Bacon, 1958). Cloth, ix+498 pp., $3.96.

A quick glance at the exterior of the textbook and a superficial journey through its pages leave one with an impression of a book with extremely pleasant physical characteristics. The format appears to be just right, the use of red color displays superb taste, and the drawings throughout the book somehow attract and hold one's attention.

As important as physical characteristics may be, the really valuable feature of any textbook is the quality of its contents.

A cursory examination reveals geometric contents similar to those found in many present-day plane geometry textbooks. Thus, a chapter on "Getting Acquainted with Geometry," in which an assortment of subjects such as the nature of plane and solid geometries, lines, circles, constructions, measurement of angles, equation axioms, and definitions are touched upon, is followed by a discussion of proofs of theorems, geometric constructions, geometry of parallel lines, circles and loci, inequalities, ratio and proportion, and similar polygons, coupled with just a taste of work in trigonometric ratios labeled as optional, culminating in the work on areas leading to the derivation of a formula for the area of a circle. The book is concluded with a chapter on "Geometric Reasoning in Everyday Life" and with a chapter on "Co-ordinate Geometry" thrown in as optional material.

A more diligent probe leads one to learn that the phrase "geometric reasoning" is of no secondary importance to the authors. They use this phrase as synonymous with "clear thinking," "straight thinking," and "logical reasoning." The claim that the book is "modern in every respect without sacrificing the straight thinking" is somewhat puzzling. Does "modernizing a book" imply an increased probability of straying from good thinking? Be that as it may, the authors do make it clear that training to do clear thinking is one of the major goals to be achieved through the study of geometry.

The student, generally, learns to think well by reading books which embody good thinking. A thoughful examination of the book reveals numerous inadequacies.

One must agree that precise definitions are a necessary part of a clear mathematical discourse. This text cannot be commended for achieving this clarity.

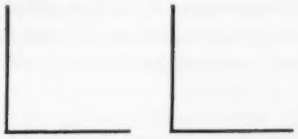
The first definition which appears in the book is that of the word "definition." It is: "Definition is a statement of the exact meaning of a term or expression." Then the authors go on to state that "a good definition should first place the term in its nearest class, or group, and then state a sufficient number of properties of the thing defined to identify it completely." The author's definition of "definition" does not achieve this. Is a class of statements the nearest class to definitions? How is one to know it? What is the meaning of "exact meaning"? Of "an expression"? The point is that the authors' language used in stating criteria for judging clarity of geometric exposition is as unclear as the language of the exposition itself.

Some other examples of lack of clarity follow:

"The name 'AB' is used to mean both a line segment (a set of points) and the length of a line segment. The student must play a game of guessing which meaning is intended in different instances."

" 'A curved line' is defined as 'a line no part of which is straight.' " What do "part" and "straight" mean? These are key words and as such they must be defined.

The statement "the sum of the angles around a point on one side of a straight line passing through the point equals two right angles, or 180°" harbors at least one major difficulty. Two right angles are two geometric objects (by the authors' definition on page 36) such as those in the accompanying figure.



How can one say that these two objects and 180° are the same thing?

The examples above are illustrative of many other similar inadequacies of this text. The inevitable conclusion is that, although the text can probably be used to good advantage in teaching much geometry, it cannot possibly contribute to the learning of habits of clear, and thus truly mathematical, thinking.—*Eugene D. Nichols, Florida State University, Tallahassee, Florida.*

• TESTING TIME

Edited by Robert S. Fouch, Florida State University, and Robert Kalin, Florida State University, Tallahassee, Florida

Mathematics as a sport

by Carl Ladd, Science Research Associates, Chicago, Illinois

Early one Saturday morning last spring, several hundred Chicago-area high school boys and girls filed into an auditorium at Northwestern University to compete in a mathematics tournament which climaxed weeks of study by the participants. The Future Engineers of America, sponsor of the tournament, hopes contests such as this will make the attainment of competence in mathematics as exciting and challenging as football and basketball are to high school students.

The Future Engineers of America is a nonprofit organization established to encourage high school students to prepare for careers in engineering, to stimulate interest in mathematics and the physical sciences, and to provide additional educational opportunities in mathematics. Its approach is a straightforward one designed to capitalize on the interests of high school students and to apply the force of organized competition through self-study and tournaments. Instructional and tournament materials contain problems built around baseball, flying saucers, and everyday problems faced by engineers and technicians. FEA's Board of Directors consists of two consulting engineers, the director of an industrial research laboratory, and the chairman of the mathematics department of a large Midwest university. The Board is assisted by an Advisory Committee composed of a high school mathematics teacher, two engineers from private industry, and a professor of mathematics. With this broad representation from high schools, colleges, and industry, FEA is thus in a position to organize and conduct a stimulating, integrated program to supplement classroom instruction in mathematics.

The backbone of FEA's instructional program is the self-study kit, which, as its name implies, is used by the student without help from teachers or other adults, although a student is encouraged to study with his friends. Prepared by a committee of teachers and engineers, the kit covers a variety of topics ranging from checking multiplication by casting out nines and elevens to the theory of sets and geometrical methods for finding slopes and areas. Of course, questions and problems that are included vary widely in difficulty because the kit is used by students in all four grades of high school. In addition to prob-

Editors' Note: Dr. Carl E. Ladd, as a member of Science Research Associates' Research Department, helped promote the New Orleans mathematics competition and the Future Engineers of America tests. Before joining SRA, Dr. Ladd was a psychometrist and instructor at the University of Kansas, where he earned his Ed.D. in 1955. Your editors are grateful to Dr. Ladd for his willingness to describe the FEA program, one of the more interesting mathematics tournaments, which seem to be growing in popularity.

lems related to the topics discussed in the kit, a number of "brain teasers" are included. Their nature is illustrated by the following examples:

1. In a Russian mathematics tournament, the following question was put: One coin out of 80 coins is false (lighter). How can you locate the false coin in four weighings on a scale with two pans without weights?
2. Freddie the Frog fell down a well at six o'clock in the morning. Every day he climbs five and two-thirds feet, and in the evening he falls asleep at 6 P.M., sleeps for twelve hours and slips back four and two-thirds feet. If the well is twenty feet deep, when will Freddie reach the top of the well?

Members of the FEA and all high schoolers are eligible to compete in the mathematics contests, of which the one held at Northwestern is typical. Students who entered the tourney vied against other students in the same grade. Separate prizes were awarded to public, parochial, and independent school students. All competitors took the same test, which was prepared by a committee similar to the one responsible for the kit. The test items paralleled closely the problems and questions in the study kit.

After completing the written test, the contestants at Northwestern were served lunch, and the scoring and processing of the answer sheets was begun. The afternoon session of the tournament was devoted to the use of the slide rule. Students familiar with the instrument participated in an hour-long slide-rule bee. Those who did not know how to use the device were given slide-rule instruction. This was followed by a slide-rule competition among seven high school teams, during which team backers urged their representatives to victory as enthusiastically as any high school cheering section. At the conclusion of the contest, the winners of the written competition were announced and prizes awarded.

What is the value of such a spirited tournament? Many scholastic competitions are probably of doubtful value, and there is considerable duplication of effort in them. These and other undesirable characteristics sometimes found in student contests led the National Association of Secondary School Principals to formulate recommended criteria for such events and to establish an approved list of contests and activities. This action has done much to improve these activities and increase their contribution to the educational program. As for the FEA tournament just described, few, if any, will criticize the objectives of the organization sponsoring it. In these critical days, the stimulation of interest and the encouragement of competence in mathematics are of the utmost importance. The testimony indicates that these objectives were, at least to some extent, accomplished. One freshman contestant said, "I liked the tournament very much and I particularly liked the self-study kits which enabled me to dig out things for myself." A girl remarked, "It was a wonderful experience. I want to go again." That these statements are indicative of the students' acceptance of and enthusiasm for this activity is supported by the results from a questionnaire sent to the students who participated in the Northwestern tournament. The FEA reported that almost 80% of those responding wanted to continue with the study-tournament program.

While these reactions were gratifying to the sponsors, an educationally sound program requires results other than mere interest or excitement. Perhaps the most significant evidence of increased student achievement through the FEA program may be found in one statistic reported by a member of the advisory committee: four questions about geometric calculus constructions were contained in the written test, and all four were answered correctly by 11% of the participants. Thus, a number of students were able to grasp and readily understand this supposedly ad-

vanced material when it was presented.

Like interscholastic athletics, interscholastic academic competitions have both strong points and weak ones, *aficionados* and critics. Whatever one's position, however, there is food for thought in the words of a Chicago *Tribune* editor who wrote of the FEA tournament, "Young people with brains can find few more entertaining and rewarding pastimes than using them. Today's schools do not induce mental fatigue in everyone enrolled in them. What could be more socially useful than for teen-agers left unexhausted by their assignments to study fresh and challenging material on their own time, and then to compete as individuals and teams?"

Have you read?

SCHENBERG, SAMUEL. "A Study of Science and Mathematics Courses Elected by the 1956 Senior Class in the Academic High Schools in New York City," *Science Education*, April 1958, pp. 225–238.

There is much publicity today about the elections students are making in the field of science and mathematics. This study, though limited to the academic high schools in New York, presents some facts that have a wider application.

It is impossible to summarize such an article, but you will note that 16 per cent of the senior classes surveyed completed four years of mathematics and two per cent, more than four years. Likewise it is interesting to note that the study of science increased the study of mathematics. Of those with two years of science, only three per cent completed three and one-half years of mathematics; of those with four years of science, 29 per cent had four years of mathematics. Fifty per cent of the seniors with four years of mathematics also had four years of science.

Information about the courses taken and the division of boys and girls is also given. You will need a little time to analyze this study, but it is well worth that time.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

SCHUSTER, CYNTHIA A. "Can We Teach the High School Student to Think?" *Educational Research Bulletin*, April 1958, pp. 90–100, 112.

Every day teachers of mathematics ask themselves the above question. In every set of objectives for teaching mathematics we find reference to this question. The author has some views on the subject.

First of all, he says, we must be aware of the fallacies in thinking, such as hasty generalizations from insufficient data, irrelevant appeals to the emotion, shifts of meanings from what is being discussed to only slightly related areas, and the use of fallacious syllogisms.

How do we prepare teachers to combat this kind of thinking? By instruction in logic and habits of thought. Develop a complete understanding of the definition and its purposes; show the nature of deductive proof as only one method; insist on a recognition that the result of an experiment is not a conclusive fact; and provide the background to show that the scientist does not believe that everything is predestined. The idea that experimentation is the only scientific method particularly needs to be corrected.

The teacher can help the student by careful analysis of the text material, discussion of axioms and postulates, and emphasis on the lives of scientists and the purposes of their conclusions. Be sure you measure the students' methods of thinking in examinations. This article will be a challenge to you.—*Philip Peak, Indiana University, Bloomington, Indiana.*

WHITEHEAD, NORTH T. "Humanism in a Scientific Age," *American Scientist*, September 1958, pp. 309–322.

In this day of satellites and atomic submarines, we are inclined to be oriented to the scientific point of view. In reality this scientific age brings about great changes in the humanistic side of our culture. The disciplines of the physical sciences will become the basis of our culture. This implies that the scientists must be cultured in the humanistic sense, but their preparation to date does not lend itself to such ends. The frame of mind of the scientist, the nature of the discipline, and the obstacle of laboratory work all are handicaps for the scientist wishing to include humanism. This transition will not be easy: teachers must open doors and develop inquiring minds. The author points up a real problem facing the world, and one we should consider in our high school classes.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

• TIPS FOR BEGINNERS

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Getting started in a ninth grade nonacademic class

by Max A. Sobel, Montclair State College, Upper Montclair, New Jersey

One of the most difficult problems faced by every teacher, new or experienced, is that of getting a class started at the outset of a term. This problem is even more difficult when one faces a nonacademic group of students with their all-too-frequent hostility toward mathematics. This problem manifests itself especially in the ninth grade, where many students enter the course with a dislike built up by eight years of failure in the elementary and junior high school grades. Before offering some concrete suggestions for handling the situation, it is advisable to examine these youths and determine their basic characteristics and needs.

High on the list of needs of adolescents is that of security. These youngsters, despite their traditional dislike of arithmetic and their lack of ability in this area, nevertheless feel a certain sense of security in the ninth grade when the subject of study is that of arithmetic. They are relieved to study that which is not new to them. At the same time, paradoxically, the rapidly changing interests of adolescents bring forth a strong desire for something new.

There is little doubt but that most "general mathematics" students are in dire need of a meaningful reteaching of arithmetic. As a matter of fact, this should probably be the fundamental task in the work of the ninth grade. On the other hand, experience indicates that beginning the ninth year with a review of arithmetic will tend to be dull, deadening, and destructive of any interest that might otherwise be aroused. It will also help strengthen in the minds of the young people the conviction that theirs is a sort of second-rate course, not on par with the ninth grade college preparatory mathematics.

The point of view being expressed here is not that we should teach anything less than mathematics, but that there are more appealing aspects of the curriculum with which to begin a new course than a review of the fundamentals of arithmetic, no matter how great the need. There follows a sample of several of these appealing topics.

None of the topics suggested are original with the author. The only faint claim to originality is the insistence that these topics have been successfully used and do make an excellent means of motivating and beginning a nonacademic class. These topics have been found to stimulate interest, develop a sense of prestige in the course, and at the same time meet the student's need for the security of the old as he simultaneously searches for new experiences. A more detailed description is given for the topic on probability and statistics, to show just how the topic may be used to introduce ninth-grade general mathematics.

1. *The use of other number bases.* It is surprising to discover how many college mathematics students and even teachers of mathematics are not familiar with methods of writing numbers in such bases as 2, 5, or 12. It is also interesting to note that many students of low ability in mathematics take to this topic and, in the course of their studies, gain an unprecedented amount of knowledge in the meaning of our decimal system of notation. Many recreations based upon the binary system of notation are excellent sources of material for motivating the nonacademic student. The use of other number bases is not a topic which can be taught in a day, but a one- or two-week unit in this area should prove fruitful.

2. *Systems of numeration.* Once again, a great deal of information and many concepts concerning the decimal system of notation can be developed by means of a study of such other systems as that of the Egyptian hieroglyphic method of numeration. Such a unit can be made especially interesting if organized to correlate with the work of the social studies class. Correlation of subject matter is one means whereby it is possible to provide the student with a sense of security.

3. *Elementary surveying.* The construction and use of a simple clinometer or hypsometer to solve field problems of indirect measurement will appeal to the practical nature of the nonacademic student, and it will at the same time provide the opportunity to offer much needed instruction in the area of measurement. The belief that "surveying" is taking place adds to the prestige of the course.

4. *Slide rule.* Although in general it is felt to be unwise to teach mechanically that which can not be developed with understanding, there is enough to be said for a unit on the slide rule so that the end justifies the means. A unit on the slide rule is another topic which will help develop prestige and at the same time will motivate computational practice and develop skills in the ability to estimate.

5. *Ancient methods of computation.* The Russian peasant method of multiplication, the galley method, finger multiplication, and others never fail to evoke interest while they at the same time motivate traditional computation.

6. *Probability and statistics.* Although this topic is currently recommended for the twelfth grade of the college preparatory program, much of the material can be developed on an intuitive basis with nonacademic students. A great deal of interest can be developed by converting the classroom into a laboratory and having the students toss pennies, roll dice, pick numbered chips from a box, etc. Thus they can gather and analyze their own real data.

Results of individual experimentation can be pooled and such mathematical concepts as ratio and per cent can be brought to light. To be specific, in tossing pennies, the students can be asked to record their data on a chart and, after each toss, record the ratio of heads and tails to the total number of tosses as well as the per cent of heads and tails. These data can then be graphed and discussed with regard to such items as fluctuation and the effect of a large number of tosses upon this fluctuation. A sample of the type of chart that can be used is shown in Table 1. The related graph is shown in Figure 1.

Figure 1

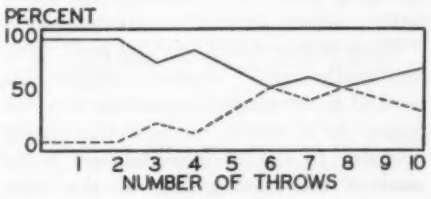


TABLE 1

CHART TO RECORD EXPERIMENTS WITH TOSSING COINS

	Number of throws									
	1	2	3	4	5	6	7	8	9	10
Heads	x	x		x			x		x	x
Tails			x		x	x		x		
Ratio of heads	1/1	2/2	2/3	3/4	3/5	3/6	4/7	4/8	5/9	6/10
Ratio of tails	0/1	0/2	1/3	1/4	2/5	3/6	3/7	4/8	4/9	4/10
Per cent of heads	100	100	67	75	60	50	57	50	56	60
Per cent of tails	0	0	33	25	40	50	43	50	44	40

The preceding development leads very nicely into a discussion of the applications of probability in a mortality table and its use in life insurance.[1]

The teacher can extend the experiment to home assignments in which students are asked to toss a specific number of coins, say four, and to record the number of heads obtained. Cumulative frequency distributions can then be made in class to compare the observed frequencies with the theoretical frequencies which might normally be expected. A similar type of laboratory approach can be done in the classroom by listing all the possible outcomes with one toss of a pair of dice. These theoretical probabilities can then be compared with real data gathered by the students. A discussion of "odds" in favor of an event can be held.

One interesting experiment is prepared by providing groups or students with bags of chips of two colors, say ten white and forty red. Students are to take samples of five, replacing their sample after each drawing, and record their data on a chart such as the following:

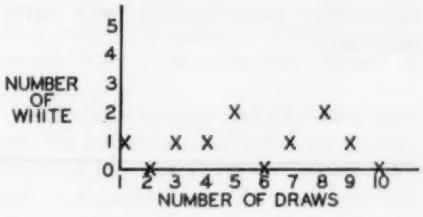


Figure 2

After taking a specified number of samples, say ten, each group is asked to estimate the per cent of white chips in the original distribution. After this, a second set of chips, of unknown distribution, is placed in the original container. Students continue to take samples and record their data on the chart shown above. Thus if the teacher were to add thirty white and twenty red chips, the new distribution would be one with forty per cent of its chips white. Again an estimate is required on the basis of a specific number of samples, as well as a decision on whether any change in the population has taken place.

[1] The Institute of Life Insurance, 488 Madison Avenue, New York, has prepared a booklet, *The Mathematics of Life Insurance*, which begins with such an approach. Copies are available in classroom quantity.

The student learns that variation is normal, but that the pattern of variation changes.

The preceding experiments, outside of their value in teaching certain elements of arithmetic and graphing, should lead to an understanding of the basic meaning of statistics as a collection and analysis of real data. In addition, the students will come to recognize that variation is normal and to be expected, a fact which may help them to understand their own rates of growth and development.

There exists also a variety of topics which may not lend themselves to lengthy units, but which may be used for a period or less merely as a means of stimulating interest and diverting attention momentarily from the tedious process of reviewing the fundamentals. In this category such items may be included as: magic squares; abacus; Roman counting board; shortcuts in computation; history of measurement; mathematical puzzles, games, recreations; paper folding; and curve stitching.

Unfortunately, no single source of reference exists for all the items previously mentioned. The wise teacher will begin to make a collection of these ideas to use them as a reference not only for the start of the term but also occasionally to add variety to the traditional course at any time.

To summarize, the devices mentioned above, none of which is original, form a sample of the many topics which are mathematical in nature and may be used to help motivate a general mathematics class. It is the conviction of the writer, borne out by experience, that a greater measure of success can be achieved with the nonacademic student if time is spent at the beginning of a new semester on something other than the traditional "two weeks' review." As was pointed out earlier in the article, this procedure seems to follow from a study of the basic needs and characteristics of adolescents, recognizing their need for the security of the old combined with their rapidly changing interests and desire for new experiences.

Said in 1911!

Until recent years, elementary algebra has been largely a miscellaneous collection of rules for the manipulation of algebraic expressions, and is not at all the developed science that elementary geometry has long since become. In fact, if it were not for the study of plane geometry in our schools, it is doubtful whether our school children would ever derive, from their study of algebra alone, any clear notion of what is meant by a mathematical demonstration.—*E. V. Huntington.*

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

You'll Be Welcome in Dallas

April 1–4, 1959

We of Dallas are glad to have this opportunity to invite every teacher of mathematics to attend the April convention of The National Council of Teachers of Mathematics in Dallas. Many of you have never been to Dallas; so why not come and see just what you have missed?

You will find an atmosphere of progress very similar to that of other large cities you have known, coupled with a spirit of hospitality unknown in most metropolitan centers. You will find beautiful buildings and tall skyscrapers—our new forty-two story Southland Life Building will be completed by April. Dallas is a leading city in the fields of banking and insurance. You may wish to take a guided tour of the Eleventh Federal Reserve Bank, or you may take a guided tour of the beautiful and spacious forty-story building of the Republic National Bank, the eighteenth largest bank in the United States. The ladies will enjoy a tour of the famed Neiman-Marcus department store.

Dallas has many cultural and religious institutions. Three denominations have their largest congregations in Dallas. You will see the Texas State Fair Park, where annually more than two million persons attend the largest state fair in the world. Here is the Cotton Bowl, where one of the classic New Year's football games is played.

Dallas is a leading transportation center. Six rail lines offer passenger service. Many of you will arrive by plane at the new, commodious Love Field Airport, one of the largest and finest to be found anywhere.

This convention is sponsored jointly by the Greater Dallas Council of Teachers of Mathematics and the Dallas Independent School District. We hope that you come early enough for the barbecue Wednesday night and stay until the last program is concluded. Nearly three hundred persons are now at work on eighteen committees intent on making your visit to Dallas an event you will always remember with satisfaction.

ARTHUR W. HARRIS and
JESSE F. CARDWELL,
Cochairmen, Committee on Arrangements

Report of the Publications Board

by H. Glenn Ayre, Western Illinois University, Macomb, Illinois

The Publications Board consists of Henry Van Engen, Editor of THE MATHEMATICS TEACHER, Ben Sultz, Editor of *The Arithmetic Teacher*, W. W. Sawyer, Editor of *The Mathematics Student Journal*, L. A. Ringenberg, Editor of Supplementary Publications, Henry Swain, Clifford Bell, and H. Glenn Ayre, Chairman.

The duties as set forth for this committee are: "To propose and co-ordinate publication policies of the Council, to make recommendations to the Board concerning major publications, and to report action on minor publications to the President." Regular meetings are held at the time of the meetings of the Council, but in the meantime, a great deal of correspondence is necessary to study proposed publications and make some recommendations about them. Much of the correspondence is concerned with material presented for supplementary publications. The non-editorial members of the Board assume some responsibility in examining manuscripts and finding referees. The work of this committee in connection with our regular publications is more or less a matter of policy because the editors assume the responsibility of finding referees for journal material and publishing the journals.

We are constantly adding to our supplementary and short publication titles. At the present time, there are three or four manuscripts under consideration. To keep up with the demand for supplementary material, it is necessary to plan a long way in advance and to keep a number of manuscripts constantly on hand. The Publications Board also assumes considerable responsibility for the publication of yearbooks. A good backlog of material is on hand and a number of yearbooks are planned and being written at the present time. We should like to encourage people interested in contributing to NCTM publications to write to one of our editors or the Publications Board.

Report of the Place of Meeting Committee

by H. Glenn Ayre, Western Illinois University, Macomb, Illinois

Two years ago President Fehr appointed the following people to serve on the Place of Meeting Committee: Marguerite Brydegaard, Forest N. Fisch, Alice Hach, James Nudelman, and H. Glenn Ayre, Chairman.

The duties of the committee were stated as follows: "To study, plan, and report to the Board concerning geographic and strategic locations of all Council meetings and to recommend to the Board a planned sequence of convention cities through 1965." Evidently the appointment of the committee was planned to include geographic distribution throughout the United States. Most of the work of the committee has been done by correspondence. Each member has consulted with affiliated groups and other people who might help in locating conventions in accessible areas well distributed throughout the United States. The chairman has been in touch with many people and reported his results to other members of the committee. The results of this correspondence have been summarized and placed before the committee members for suggestions and revisions. Direct discussion has been limited to the regular meetings of the National Council.

The committee feels that it should collect information and make recommendations to the Board of Directors. Final decision of convention locations rests with the Board. At the present time, the Board has approved convention locations for the springs of 1959, 1960, 1961, and summer conventions for 1959, 1960.

At the 1958 Christmas meeting of the Council, the Official Board approved recommendations for convention locations for spring and summer meetings through 1963 and for one Christmas meeting. The committee is considering invitations for conventions to be scheduled for 1964 and 1965.

The committee wishes to extend appreciation especially to affiliated groups who have taken the responsibility for organizing groups to sponsor conventions and be responsible for local arrangements. Besides affiliated groups, many leaders in the National Council have assumed major responsibility for the location of NCTM conventions. The committee would be glad to hear from persons or groups of persons who are interested in sponsoring one of our annual meetings—either the spring meeting or the summer meeting.

Report of the Nominating Committee

Miss Irene Sauble is a nominee for the Board of Directors. The biographical material for Miss Sauble was inadvertently left out of the January issue.

IRENE SAUBLE

Irene Sauble

North Central Region

Director of Exact Sciences, Detroit Public Schools, Detroit, Michigan.

A.B., University of Michigan; M.A., University of California.

Teacher of Mathematics, Northwestern High School, Detroit, Michigan; Supervisor of Mathematics and Director of Exact Sciences, Detroit Public Schools; Part-Time Associate Professor of Teaching of Mathematics, Wayne State University, Detroit, 1936–51; Instructor during summer sessions, Claremont College, Claremont, California, 1931, Loyola University, Chicago, 1932, Chico State College, Chico, California, 1933; Served on staff of workshops in Oakland, California, 1950–51.

Member: NCTM; NEA; Michigan Council of Teachers of Mathematics; Detroit Council of Teachers of Mathematics; Michigan Education Association; Phi Beta Kappa

Activities in NCTM: Vice-president, elementary school level, 1952–54; Program chairman for 1953 summer meeting; Member of the Elementary Curriculum Committee, 1955–58; North Central Regional Representative for Affiliated Groups, 1954–56; Member, Twenty-fifth Yearbook Committee, 1956–59

Publications: "The Enrichment of the Arithmetic Course—Utilizing Supplementary Materials and Devices" (Sixteenth Yearbook of the NCTM), "Teaching Fractions, Decimals and Percent: Practical Applications" (University of Chicago Monograph, No 63), "Applications of the Film in Mathematics," and "Development of Ability to Estimate and Compute Mentally" (*The Arithmetic Teacher*); Co-author of "The Measurement of Understanding of Elementary School Mathematics," "Reading in the Content Fields," and "Arithmetic We Need," Grades 3–8.

Letters to the editors

Dear Editor:

Thank you for the excellent book reviews often found in THE MATHEMATICS TEACHER.

Purely descriptive reviews are of little use to one who must buy sight unseen as a general rule. Critical, although not hypercritical, reviews are most helpful to the librarian seeking new titles for school use. Too often one is offered a virtual reprint with sufficient new material to secure a current copyright—ranging from a paragraph to a chapter or two. Well-known names, authors and publishers, are not the invariable sign of a worthy addition to the shelves. Professional courtesy in excess between authors, the reviewer and the reviewed, can mislead the prospective purchaser.

Thank you again for this feature of your journal!

Yours truly,
JOHN B. MCGRAW, JR.,
West Virginia Institute of Technology
Montgomery, West Virginia

My dear Editor:

This is an account of how our local branch of the National Council is trying to meet the situation of many teachers in secondary schools attempting to teach Modern Mathematics concepts. We are not an organized class, but we do follow a monthly outline of work on mimeographed sheets. Teachers in allied fields have asked to "sit in" on the discussions so we feel we are filling a need.

The Pinellas County Council of Mathematics Teachers found that it was unable to lure any Modern Mathematics extension course from Florida State University or the University of Florida to St. Petersburg. The teachers, especially those in the senior high schools, felt they wanted to know more about the content and methods of Modern Mathematics than each could get individually from books. They therefore have a group discussion once a month led by the various members of the St. Petersburg Junior College mathematics faculty. The teachers meet in a classroom of the junior college and use as a background text Allendoerfer and Oakley's *Principles of Mathematics*. There are no "credit" hours for this work, but the transition to the use of new texts is easier than it would have been otherwise, and the secondary school students in Pinellas County benefit from the introduction of the concepts of elementary set theory, basic logic and functions, both algebraic and trigonometric. The "Modern Mathematics" approach has been introduced in the junior college freshman course for engineers and mathematics majors, and beginning with the second semester of this college year, trigonometry will be so taught. We find the high school teachers eager to give their time, without any material gain, to know this new field.

Yours truly,
CAROL S. SCOTT,
Chairman Mathematics Department,
St. Petersburg Junior College
St. Petersburg, Florida

Dear Editor:

A student of mine, Mr. Robert M. Lloyd of Fort Pierce, Florida, has worked out an interesting problem which I think might be worth publishing in THE MATHEMATICS TEACHER. The problem and proof are original with Mr. Lloyd, who is a member of the freshman class at Furman.

Yours truly,
JAMES H. KELLER
Furman University
Greenville, South Carolina

Problem: Given a parabolic curve $y=ax^2+bx+c$ to prove that the square of the distance from the vertex of the parabola to the origin minus the square of the distance from the vertex to either of the points $(r_1,0)$ or $(r_2,0)$ equals c/a where r_1 and r_2 are the roots of $ax^2+bx+c=0$.
Proof: The roots of the equation $0=ax^2+bx+c$ are $-b/2a+(b^2-4ac)^{1/2}/2a$ and $-b/2a-(b^2-4ac)^{1/2}/2a$. The vertex of the parabola is at $-b/2a$, $(-b^2+4ac)/4a$. Using the distance formula, we find the square of the distance from the vertex to the origin to be

$$(4b^2+b^4-8ab^2c+16a^2c^2)/16a^2.$$

The square of the distance from the vertex to either $(r_1,0)$ or $(r_2,0)$ equals

$$(4b^2-16ac+b^4-8ab^2c+16a^2c)/16a^2.$$

Subtracting the last fraction from the one just above gives the desired result, c/a.

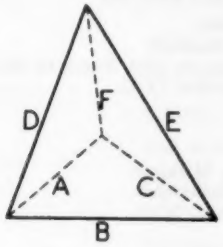
Dear Sir:

It seems that Mr. Oesterle in his article, "A note concerning the construction of $\sqrt{n}$," became so overjoyed with his perhaps unique theoretical approach to the problem at hand that he is unaware that $\sqrt{7^2+3^2}=\sqrt{58}$, not $\sqrt{59}$ as he would have us believe!

Sincerely,
JOHN BARRY LOVE
Asst. Professor of Mathematics
Eastern Baptist College
St. David, Pa.

Dear Prof. Van Engen:

This summer I had an inquiry about the formula for the volume of a tetrahedron in terms of its six edges. Since I have never seen this formula in print, I am sending it along in case you think it might be of interest to some of the readers of THE MATHEMATICS TEACHER. Its derivation would be a good project for the better students. I have a hunch the expression under the radical might be factorable, but I haven't spent much time looking at this phase of it.



If the six edges of a tetrahedron are a, b, c, d, e, and f and oriented as shown in the accompanying figure, then the volume of the tetrahedron is given by

$$V=\frac{1}{12}\,(a^2b^2e^2+a^2b^2f^2+a^2c^2d^2+a^2c^2e^2+a^2d^2e^2$$
$$+a^2e^2f^2+b^2c^2d^2+b^2c^2f^2+b^2d^2f^2+b^2e^2f^2+c^2d^2e^2$$
$$+c^2d^2f^2-a^2b^2c^2-a^2d^2f^2-b^2d^2e^2-c^2e^2f^2-a^4e^2$$
$$-a^2e^4-b^4f^2-b^2f^4-c^4d^2-c^2d^4)^{1/2}.$$

Very truly yours,
JOHN P. HOYT
Professor of Mathematics
U. S. Naval Academy
Annapolis, Maryland

Dear Sir:

It was interesting to note an amusing coincidence in the October 1958 MATHEMATICS TEACHER which arrived in the mail today.

On page 443 we find Lewis Carroll's reference to a famous paradox about a race between a hare and a tortoise. If I remember correctly the argument went like this:

> At the outset, the tortoise is in the lead and the hare trying to overtake. In order to overtake the tortoise, the hare must first reach the position held by the tortoise now, However, by the time the hare reaches that position, the tortoise will then occupy a new position ahead of the hare. The argument can then be repeated. Conclusion: Since there is no end to this, it is impossible for the hare to overtake the tortoise.

Then on page 453 we find the same fallacy offered as "Micky's proof of the medians theorem." We can all admire any 14-year-old who can formulate such an argument—however, we should not lose sight of the fact that it is fallacious.

Mr. Keedy stated—without careful consideration, no doubt—that "with a bit of doing . . . it (could) be made rigorous." A simple counterexample shows this to be impossible.

In any triangle, pick at random a point on each of two of the sides, and join each to its opposite vertex. Then pick a point on the third side in such a way that when it is joined to its opposite vertex, the three lines do *not* concur. Now apply Micky's argument repeatedly. "Since there is no end to this," the three (nonconcurring) lines "have to meet in a point."

Yours agreeably,
WILLIAM KOENEN
St. Paul, Minnesota

Dear Editor:

Probably by now you have had a flood of letters pointing out that "Micky's proof" of the median theorem (October 1958, p. 453) is no proof at all. By precisely the same reasoning one can "prove" that the lines joining the vertices of a triangle to *any* three random points on the respective opposite (unproduced) sides are concurrent, which is obviously false.

It is a pity that such an error was allowed to slip by, because the idea of the paper—the sponsoring of student creativity—is vitally important today. The author's admirable thesis will tend to be discredited by the unfortunate choice of illustration.

Sincerely yours,
C. STANLEY OGILVY
Hamilton College
Clinton, New York

Author's comment on Professor Ogilvy's letter:

Professor Ogilvy's point that Micky's "proof" is no proof at all is well taken. Micky's "proof" is not an argument which is convincing to persons who are mathematically mature. However, it is appropriate to remember that the event described was a first success (and a spectacular one) for a youngster who has scarcely begun to mature mathematically. It is not unwise, either, in teaching mathematics to keep in mind that the degree of rigor required in a proof to make it convincing varies with the maturity of the entire mathematical community. The real test of a proof is whether or not it is convincing to one's colleagues.

To Micky and *his* peers the argument about medians was convincing. Therefore in a very real sense it was indeed a proof. The principal objection to it is, of course, that it contains nothing explicit about the fact that the nested sequence of triangles has a point for its limit. To be sure, this question should be posed to a youngster so that he can sharpen his argument and mature further. But that is a sequel to the present story.—M. L. KEEDY.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

THIRTY-SEVENTH ANNUAL MEETING

April 1–4, 1959
Baker Hotel, Dallas, Texas
Arthur W. Harris, 4701 Cole Avenue, Dallas 5, Texas

JOINT MEETING WITH NEA

June 29, 1959
St. Louis, Missouri
M. H. Ahrendt, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NINETEENTH SUMMER MEETING

August 17–19, 1959
University of Michigan, Ann Arbor, Michigan
Phillip S. Jones, Mathematics Department, University of Michigan, Ann Arbor, Michigan

Other professional dates

Mathematics Section of the New York Society for the Experimental Study of Education

February 14; March 14; April 17, 1959
Teachers College, Columbia University, New York, New York
John A. Schumaker, Montclair State College, Montclair, New Jersey

Illinois Council of Teachers of Mathematics

March 7, 1959, Greenville, Illinois
March 28, 1959, Macomb, Illinois
April 11, 1959, Normal Illinois
April 15, 1959, Charleston, Illinois
April 18, 1959, Arlington Heights, Illinois
April 19, 1959, Carbondale, Illinois
T. E. Rine, Illinois State Normal University, Normal, Illinois

Chicago Elementary Teachers' Mathematics Club

March 9, 1959
Toffenetti's Restaurant, 65 W. Monroe Street, Chicago, Illinois
Romana H. Goldblatt, Burley School, Chicago, Illinois

This is the essence of set theory: It considers not only the sequence of numbers but also the totality of its subsets as a closed aggregate of objects existing in themselves. In this sense it is based on the actually infinite. But once this has been accepted, the vast structure of analysis has an unshakeable firmness; it is securely founded, in all its parts based on sound argument, exact in its concepts, without gaps in its proofs. It has thus gained a foundation which guarantees the unconditional intersubjective agreement of all workers in its field.—*Hermann Weyl*, Philosophy of Mathematics and Natural Science.

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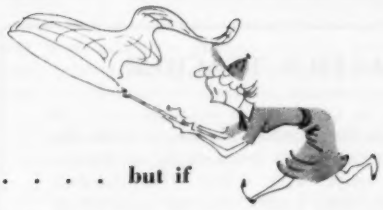

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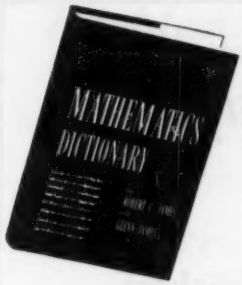

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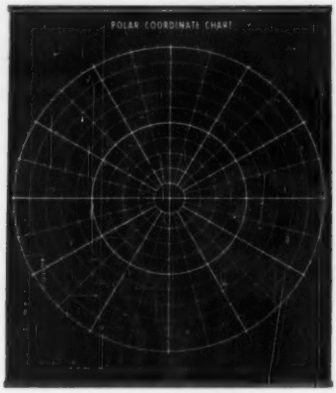

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